Carnegie Mellon

School of Computer Science

Deep Reinforcement Learning and Control

# Function Approximation in RL

Fall 2021, CMU 10-703

Instructors
Katerina Fragkiadaki
Russ Salakhutdinov

# Used Materials

- Disclaimer: Much of the material and slides for this lecture were borrowed from Rich Sutton's class and David Silver's class on Reinforcement Learning.

# Large-Scale Reinforcement Learning

- In problems with large number of states, e.g.
  - Backgammon: 10^20 states
  - Go: 10^170 states
  - Helicopter: continuous state space

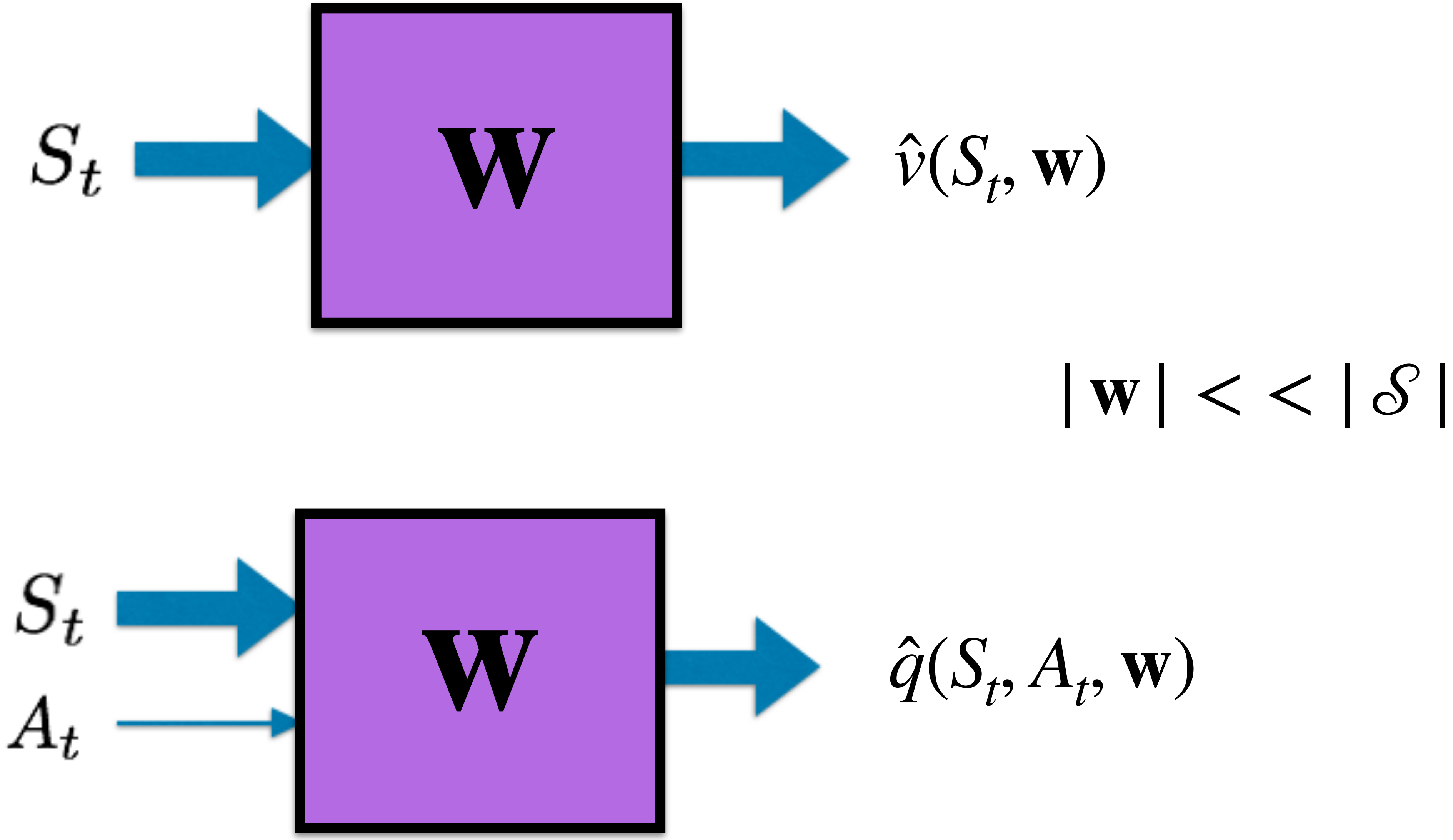
tabular methods that enumerate every single state do not work.

# Value Function Approximation (VFA)

- So far we have represented value function by a lookup table
  - Every state s has an entry $V(s)$, or
  - Every state-action pair $(s, a)$ has an entry $Q(s, a)$
- Problem with large MDPs:
  - There are too many states and/or actions to store in memory
  - It is too slow to learn the value of each state individually
- Solution for large MDPs:
  - Estimate value function with function approximation
    $\hat{v}(s, \mathbf{w}) \approx v_\pi(s)$ or $\hat{q}(s, a, \mathbf{w}) \approx q_\pi(s, a)$
  - *Generalize* from seen states to unseen states

# Value Function Approximation (VFA)

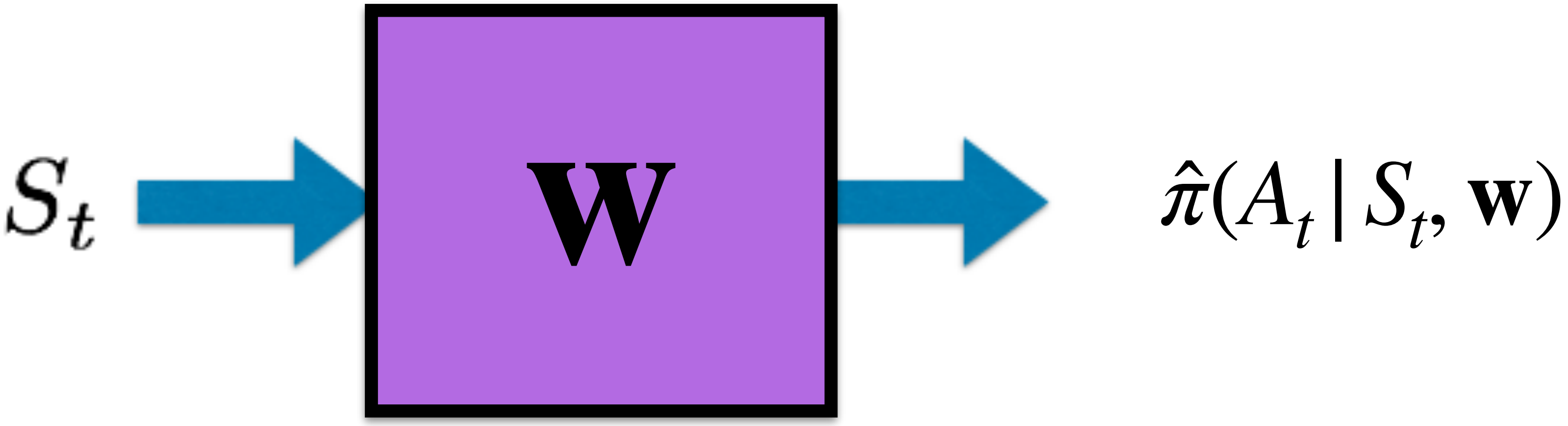
- Value function approximation (VFA) replaces the table with a general parameterized form:



When we update the parameters $\mathbf{w}$, the values of many states change simultaneously!

# Policy Approximation

- Policy approximation replaces the table with a general parameterized form:



# Which Function Approximation?

- There are many function approximators, e.g.
    - Linear combinations of features
    - Neural networks
    - Decision tree
    - Nearest neighbour
    - Fourier / wavelet bases
    - ...

# Which Function Approximation?

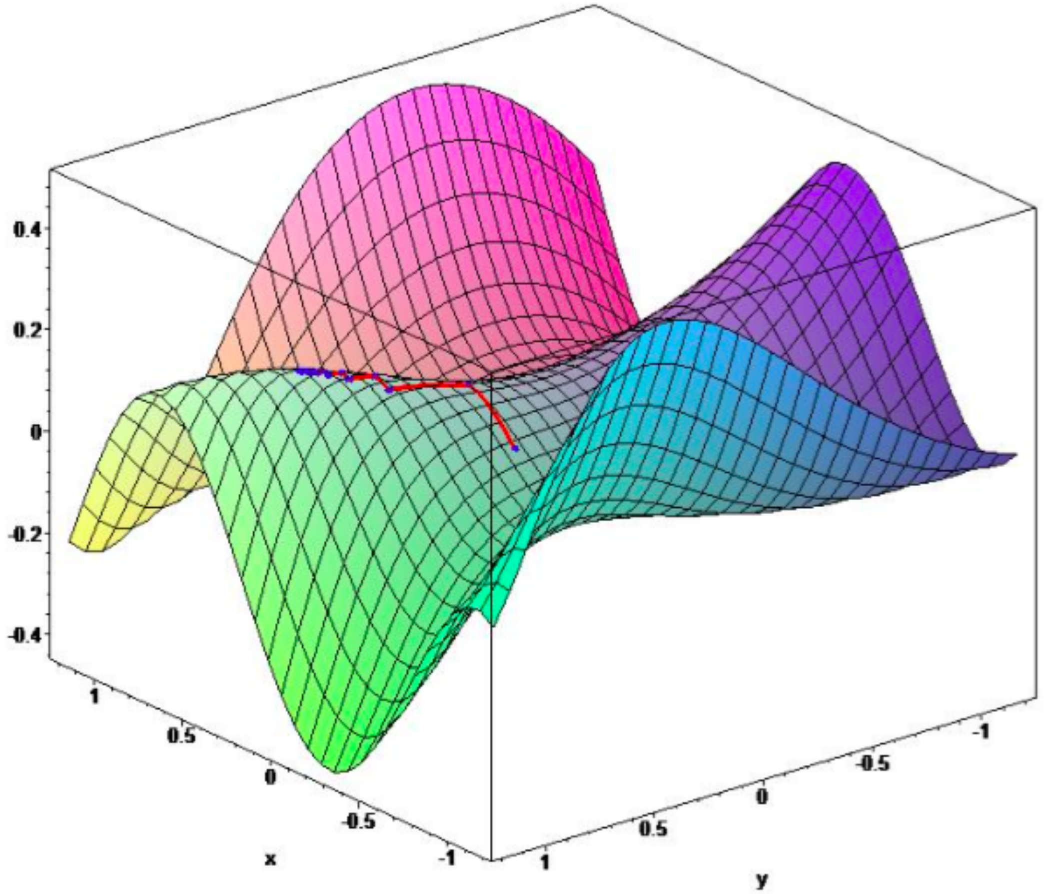
- There are many function approximators, e.g.
    - Linear combinations of features
    - Neural networks
    - Decision tree
    - Nearest neighbour
    - Fourier / wavelet bases
    - ...

- differentiable function approximators

# Gradient Descent

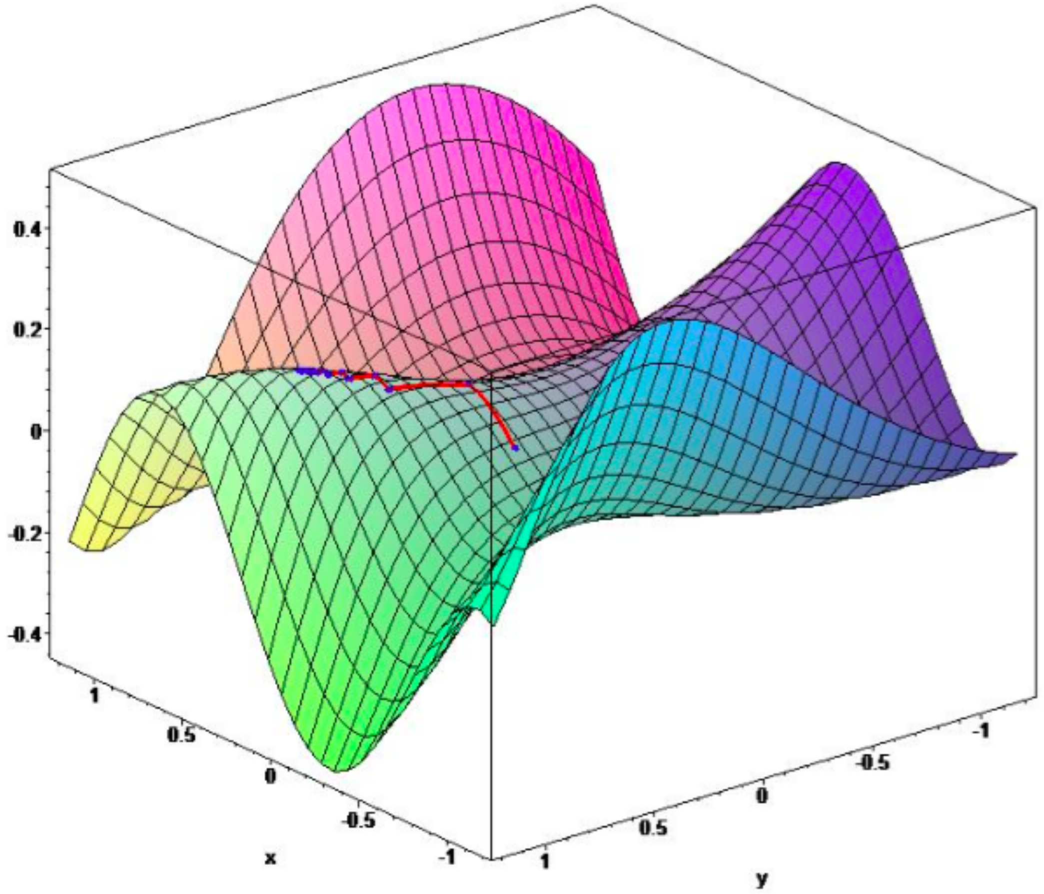
- Let J(w) be a differentiable function of parameter vector w
- Define the gradient of J(w) to be:

$$\nabla_{\mathbf{w}} J(\mathbf{w}) = \begin{pmatrix} \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}_1} \\ \vdots \\ \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}_n} \end{pmatrix}$$



# Gradient Descent

- Let J(w) be a differentiable function of parameter vector w
- Define the gradient of J(w) to be:



$$\nabla_{\mathbf{w}} J(\mathbf{w}) = \begin{pmatrix} \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}_1} \\ \vdots \\ \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}_n} \end{pmatrix}$$

- To find a local minimum of J(w), adjust w in direction of the negative gradient:

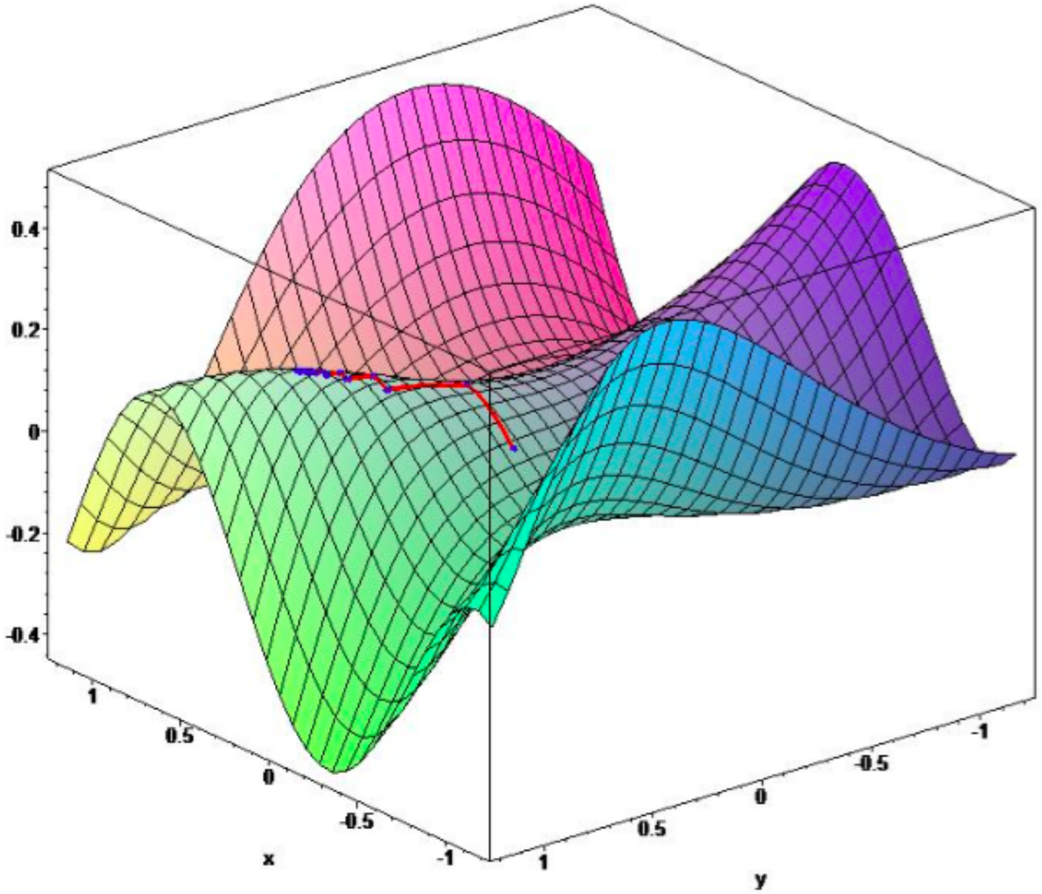
$$\Delta \mathbf{w} = -\frac{1}{2}\alpha \nabla_{\mathbf{w}} J(\mathbf{w})$$

Step-size

# Gradient Descent

- Let J(w) be a differentiable function of parameter vector w
- Define the gradient of J(w) to be:

$$\nabla_{\mathbf{w}} J(\mathbf{w}) = \begin{pmatrix} \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}_1} \\ \vdots \\ \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}_n} \end{pmatrix}$$



- Starting from a guess $\mathbf{w}_0$
- We consider the sequence $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \ldots$

  s.t. : $\mathbf{w}_{n+1} = \mathbf{w}_n - \frac{1}{2}\alpha \nabla_{\mathbf{w}} J(\mathbf{w}_n)$
- We then have $J(\mathbf{w}_0) \geq J(\mathbf{w}_1) \geq J(\mathbf{w}_2) \geq \ldots$

# Our objective

- Goal: find parameter vector w minimizing mean-squared error between the true value function $v_\pi(S)$ and its approximation $\hat{v}(S, \mathbf{w})$:

$$J(\mathbf{w}) = \mathbb{E}_\pi\left[\left(v_\pi(S) - \hat{v}(S, \mathbf{w})\right)^2\right]$$

# Our objective

- Goal: find parameter vector w minimizing mean-squared error between the true value function $v_\pi(S)$ and its approximation $\hat{v}(S, \mathbf{w})$:

$$J(\mathbf{w}) = \mathbb{E}_\pi\left[\left(v_\pi(S) - \hat{v}(S, \mathbf{w})\right)^2\right]$$

- Let $\mu(S)$ denote how much time we spend in each state $s$ under policy $\pi$, then:

$$J(w) = \sum_{n=1}^{|\mathcal{S}|} \mu(S)\left[v_\pi(S) - \hat{v}(S, \mathbf{w})\right]^2 \quad \sum_{s \in \mathcal{S}} \mu(S) = 1$$

- Very important choice: it is OK if we cannot learn the value of states we visit very few times, there are too many states, I should focus on the ones that matter: the RL solution to curse of dimensionality.

# Our objective

- Goal: find parameter vector w minimizing mean-squared error between the true value function $v_\pi(S)$ and its approximation $\hat{v}(S, \mathbf{w})$:

$$J(\mathbf{w}) = \mathbb{E}_\pi \left[ \left( v_\pi(S) - \hat{v}(S, \mathbf{w}) \right)^2 \right]$$

- Let $\mu(S)$ denote how much time we spend in each state $s$ under policy $\pi$, then:

$$J(w) = \sum_{n=1}^{|\mathcal{S}|} \mu(S) \left[ v_\pi(S) - \hat{v}(S, \mathbf{w}) \right]^2 \quad \sum_{s \in \mathcal{S}} \mu(S) = 1$$

- In contrast to:

$$J_2(w) = \frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} \left[ v_\pi(S) - \hat{v}(S, \mathbf{w}) \right]^2$$

# On-policy state distribution

Let $h(s)$ be the initial state distribution, i.e, the probability that an episode starts at state $s$.

Then the un-normalized on-policy state probability  satisfies the following recursions:

$$\eta(s) = h(s) + \sum_{\bar{s}} \eta(\bar{s}) \sum_{a} \pi(a \,|\, \bar{s}) p(s \,|\, \bar{s}, a), \forall s \in \delta$$

$$\mu(s) = \frac{\eta(s)}{\sum_{s'} \eta(s')}, \quad \forall s \in \mathcal{S}$$

# Our objective

- Goal: find parameter vector w minimizing mean-squared error between the true value function $v_\pi(S)$ and its approximation $\hat{v}(S, \mathbf{w})$:

$$J(\mathbf{w}) = \mathbb{E}_\pi\left[\left(v_\pi(S) - \hat{v}(S, \mathbf{w})\right)^2\right]$$

$$\begin{aligned}\Delta\mathbf{w} &= -\frac{1}{2}\alpha\,\nabla_{\mathbf{w}}J(\mathbf{w}) \\ &= \alpha\mathbb{E}_\pi\left[\left(v_\pi(S) - \hat{v}(S, \mathbf{w})\right)\nabla_{\mathbf{w}}\hat{v}(S, \mathbf{w})\right]\end{aligned}$$

# Our objective

- Goal: find parameter vector w minimizing mean-squared error between the true value function $v_\pi(S)$ and its approximation $\hat{v}(S, \mathbf{w})$:

$$J(\mathbf{w}) = \mathbb{E}_\pi\left[\left(v_\pi(S) - \hat{v}(S, \mathbf{w})\right)^2\right]$$

$$\begin{aligned}\Delta \mathbf{w} &= -\frac{1}{2}\alpha \nabla_{\mathbf{w}} J(\mathbf{w}) \\ &= \alpha \mathbb{E}_\pi\left[\left(v_\pi(S) - \hat{v}(S, \mathbf{w})\right) \nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w})\right]\end{aligned}$$

- Starting from a guess $w_0$

# Our objective

- Goal: find parameter vector w minimizing mean-squared error between the true value function $v_\pi(S)$ and its approximation $\hat{v}(S, \mathbf{w})$:

$$J(\mathbf{w}) = \mathbb{E}_\pi\left[\left(v_\pi(S) - \hat{v}(S, \mathbf{w})\right)^2\right]$$

$$\begin{aligned}\Delta\mathbf{w} &= -\frac{1}{2}\alpha\nabla_{\mathbf{w}}J(\mathbf{w}) \\ &= \alpha\mathbb{E}_\pi\left[\left(v_\pi(S) - \hat{v}(S, \mathbf{w})\right)\nabla_{\mathbf{w}}\hat{v}(S, \mathbf{w})\right]\end{aligned}$$

- Starting from a guess $w_0$
- We consider the sequence $w_0, w_1, w_2, \ldots$ s.t. : $w_{n+1} = w_n - \frac{1}{2}\alpha\nabla_w J(w_n)$
- We then have $J(w_0) \geq J(w_1) \geq J(w_2) \geq \ldots$

# Gradient Descent

- Goal: find parameter vector w minimizing mean-squared error between the true value function $v_\pi(S)$ and its approximation $\hat{v}(S, \mathbf{w})$:

$$J(\mathbf{w}) = \mathbb{E}_\pi\left[\left(v_\pi(S) - \hat{v}(S, \mathbf{w})\right)^2\right]$$

- Gradient descent finds a local minimum:

$$\begin{aligned}\Delta\mathbf{w} &= -\frac{1}{2}\alpha\nabla_{\mathbf{w}}J(\mathbf{w}) \\ &= \alpha\mathbb{E}_\pi\left[\left(v_\pi(S) - \hat{v}(S, \mathbf{w})\right)\nabla_{\mathbf{w}}\hat{v}(S, \mathbf{w})\right]\end{aligned}$$

# Gradient Descent

- Goal: find parameter vector w minimizing mean-squared error between the true value function $v_\pi(S)$ and its approximation $\hat{v}(S, \mathbf{w})$:
$$J(\mathbf{w}) = \mathbb{E}_\pi \left[ \left( v_\pi(S) - \hat{v}(S, \mathbf{w}) \right)^2 \right]$$
- Gradient descent finds a local minimum:
$$\begin{aligned} \Delta \mathbf{w} &= -\frac{1}{2} \alpha \nabla_{\mathbf{w}} J(\mathbf{w}) \\ &= \alpha \mathbb{E}_\pi \left[ \left( v_\pi(S) - \hat{v}(S, \mathbf{w}) \right) \nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w}) \right] \end{aligned}$$
- Stochastic gradient descent (SGD) samples the gradient:
$$\Delta \mathbf{w} = \alpha \left( v_\pi(S) - \hat{v}(S, \mathbf{w}) \right) \nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w})$$

# SGD with Experience Replay

- Given experience consisting of ⟨state, value⟩ pairs

$$\mathrm{D} = \left\{ \left\langle s_1, v_1^\pi \right\rangle, \left\langle s_2, v_2^\pi \right\rangle, \ldots, \left\langle s_T, v_T^\pi \right\rangle \right\}$$

- Repeat
  - Sample state, value from experience

$$\langle s, v^\pi \rangle \sim \mathrm{D}$$

  - Apply stochastic gradient descent update

$$\Delta \mathbf{w} = \alpha \left( v^\pi - \hat{v}(s, \mathbf{w}) \right) \nabla_{\mathbf{w}} \hat{v}(s, \mathbf{w})$$

- Converges to least squares solution

# Feature Vectors

- Represent state by a feature vector

$$\mathbf{x}(S) = \begin{pmatrix} \mathbf{x}_1(S) \\ \vdots \\ \mathbf{x}_n(S) \end{pmatrix}$$

- For example
  - Distance of robot from landmarks
  - Trends in the stock market
  - Piece and pawn configurations in chess

# Linear Value Function Approximation (VFA)

- Represent value function by a linear combination of features

$$\hat{v}(S, \mathbf{w}) = \mathbf{x}(S)^\top \mathbf{w} = \sum_{j=1}^{n} \mathbf{x}_j(S)\mathbf{w}_j$$

- Objective function is quadratic in parameters $w$

$$J(\mathbf{w}) = \mathbb{E}_\pi \left[ \left( v_\pi(S) - \mathbf{x}(S)^\top \mathbf{w} \right)^2 \right]$$

- Update rule is particularly simple

$$\nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w}) = \mathbf{x}(S)$$

$$\Delta \mathbf{w} = \alpha \left( v_\pi(S) - \hat{v}(S, \mathbf{w}) \right) \mathbf{x}(S)$$

- Update = step-size × prediction error × feature value
- Later, we will look at the neural networks as function approximators.

# Incremental Prediction Algorithms

- We have assumed the true value function $v_\pi(s)$ is given by a supervisor
- But in RL there is no supervisor, only rewards
- In practice, we substitute a target for $v_\pi(s)$
- For MC, the target is the return $G_t$

$$\Delta \mathbf{w} = \alpha \left( G_t - \hat{v}\left(S_t, \mathbf{w}\right) \right) \nabla_{\mathbf{w}} \hat{v}\left(S_t, \mathbf{w}\right)$$

- For TD(0), the target is the TD target: $R_{t+1} + \gamma \hat{v}\left(S_{t+1}, \mathbf{w}\right)$

$$\Delta \mathbf{w} = \alpha \left( R_{t+1} + \gamma \hat{v}\left(S_{t+1}, \mathbf{w}\right) - \hat{v}\left(S_t, \mathbf{w}\right) \right) \nabla_{\mathbf{w}} \hat{v}\left(S_t, \mathbf{w}\right)$$

# Monte Carlo with VFA

- Return $G_t$ is an unbiased, noisy sample of true value $v_\pi(S_t)$
- Can therefore apply supervised learning to "training data":

$$\left\langle S_1, G_1 \right\rangle, \left\langle S_2, G_2 \right\rangle, \ldots, \left\langle S_T, G_T \right\rangle$$

- For example, using linear Monte-Carlo policy evaluation:

$$\Delta \mathbf{w} = \alpha \left( G_t - \hat{v}\left(S_t, \mathbf{w}\right) \right) \nabla_{\mathbf{w}} \hat{v}\left(S_t, \mathbf{w}\right)$$

- Monte-Carlo evaluation converges to a local optimum

# Monte Carlo with VFA

**Gradient Monte Carlo Algorithm for Approximating $\hat{v} \approx v_\pi$**

```
Input: the policy $\pi$ to be evaluated
Input: a differentiable function $\hat{v} : \mathcal{S} \times \mathbb{R}^n \rightarrow \mathbb{R}$

Initialize value-function weights $\boldsymbol{\theta}$ as appropriate (e.g., $\boldsymbol{\theta} = \mathbf{0}$)
Repeat forever:
    Generate an episode $S_0, A_0, R_1, S_1, A_1, \ldots, R_T, S_T$ using $\pi$
    For $t = 0, 1, \ldots, T-1$:
        $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha \big[ G_t - \hat{v}(S_t,\boldsymbol{\theta}) \big] \nabla \hat{v}(S_t,\boldsymbol{\theta})$
```

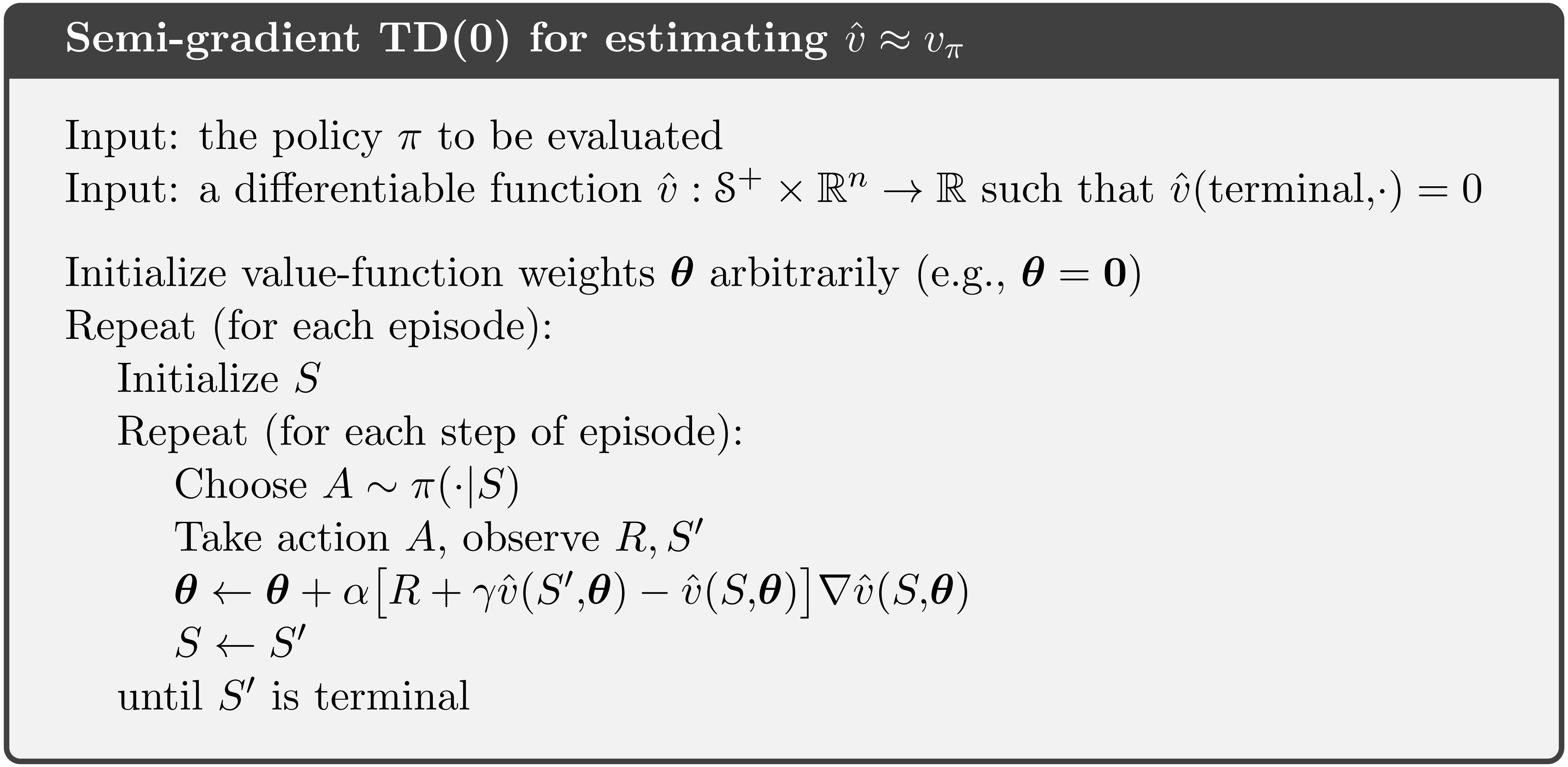
# TD Learning with VFA

- The TD-target $R_{t+1} + \gamma \hat{v}\left(S_{t+1}, \mathbf{w}\right)$ is a biased sample of true value $v_\pi(S_t)$
- Can still apply supervised learning to "training data":
  $$\left\langle S_1, R_2 + \gamma \hat{v}\left(S_2, \mathbf{w}\right)\right\rangle, \left\langle S_2, R_3 + \gamma \hat{v}\left(S_3, \mathbf{w}\right)\right\rangle, \ldots, \left\langle S_{T-1}, R_T\right\rangle$$
- For example, using linear TD(0):
  $$\Delta \mathbf{w} = \alpha \left(R + \gamma \hat{v}\left(S', \mathbf{w}\right) - \hat{v}(S, \mathbf{w})\right) \nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w})$$

We ignore the dependence of the target on $w$!

We call it semi-gradient methods

# TD Learning with VFA



**Semi-gradient TD(0) for estimating $\hat{v} \approx v_\pi$**

```
Input: the policy $\pi$ to be evaluated
Input: a differentiable function $\hat{v} : \mathcal{S}^+ \times \mathbb{R}^n \to \mathbb{R}$ such that $\hat{v}(\text{terminal},\cdot) = 0$

Initialize value-function weights $\boldsymbol{\theta}$ arbitrarily (e.g., $\boldsymbol{\theta} = \mathbf{0}$)
Repeat (for each episode):
    Initialize $S$
    Repeat (for each step of episode):
        Choose $A \sim \pi(\cdot|S)$
        Take action $A$, observe $R, S'$
        $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha\big[R + \gamma\hat{v}(S',\boldsymbol{\theta}) - \hat{v}(S,\boldsymbol{\theta})\big]\nabla\hat{v}(S,\boldsymbol{\theta})$
        $S \leftarrow S'$
    until $S'$ is terminal
```

# Control with VFA

- Policy evaluation Approximate policy evaluation: $\hat{q}(\cdot, \cdot, \mathbf{w}) \approx q_\pi$
- Policy improvement ε-greedy policy improvement

# Action-Value Function Approximation

- Approximate the action-value function

$$\hat{q}(S, A, \mathbf{w}) \approx q_\pi(S, A)$$

- Minimize mean-squared error between the true action-value function $q_\pi(S, A)$ and the approximate action-value function:

$$J(\mathbf{w}) = \mathbb{E}_\pi\left[\left(q_\pi(S, A) - \hat{q}(S, A, \mathbf{w})\right)^2\right]$$

- Use stochastic gradient descent to find a local minimum

$$-\frac{1}{2}\nabla_{\mathbf{w}} J(\mathbf{w}) = \left(q_\pi(S, A) - \hat{q}(S, A, \mathbf{w})\right)\nabla_{\mathbf{w}}\hat{q}(S, A, \mathbf{w})$$

$$\Delta\mathbf{w} = \alpha\left(q_\pi(S, A) - \hat{q}(S, A, \mathbf{w})\right)\nabla_{\mathbf{w}}\hat{q}(S, A, \mathbf{w})$$

# Linear Action-Value Function Approximation

- Represent state and action by a feature vector

$$\mathbf{x}(S,A) = \begin{pmatrix} \mathbf{x}_1(S,A) \\ \vdots \\ \mathbf{x}_n(S,A) \end{pmatrix}$$

- Represent action-value function by linear combination of features

$$\hat{q}(S,A,\mathbf{w}) = \mathbf{x}(S,A)^\top \mathbf{w} = \sum_{j=1}^{n} \mathbf{x}_j(S,A)\mathbf{w}_j$$

- Stochastic gradient descent update

$$\nabla_{\mathbf{w}} \hat{q}(S,A,\mathbf{w}) = \mathbf{x}(S,A)$$

$$\Delta \mathbf{w} = \alpha \left( q_\pi(S,A) - \hat{q}(S,A,\mathbf{w}) \right) \mathbf{x}(S,A)$$

# Incremental Control Algorithms

- Like prediction, we must substitute a target for $q_\pi(S, A)$
- For MC, the target is the return $G_t$

$$\Delta \mathbf{w} = \alpha \left( G_t - \hat{q}\left(S_t, A_t, \mathbf{w}\right) \right) \nabla_{\mathbf{w}} \hat{q}\left(S_t, A_t, \mathbf{w}\right)$$

- For TD(0), the target is the TD target: $R_{t+1} + \gamma Q\left(S_{t+1}, A_{t+1}\right)$

$$\Delta \mathbf{w} = \alpha \left( R_{t+1} + \gamma \hat{q}\left(S_{t+1}, A_{t+1}, \mathbf{w}\right) - \hat{q}\left(S_t, A_t, \mathbf{w}\right) \right) \nabla_{\mathbf{w}} \hat{q}\left(S_t, A_t, \mathbf{w}\right)$$

# Incremental Control Algorithms

**Episodic Semi-gradient Sarsa for Estimating $\hat{q} \approx q_*$**

```
Input: a differentiable function $\hat{q} : \mathcal{S} \times \mathcal{A} \times \mathbb{R}^n \to \mathbb{R}$

Initialize value-function weights $\boldsymbol{\theta} \in \mathbb{R}^n$ arbitrarily (e.g., $\boldsymbol{\theta} = \mathbf{0}$)
Repeat (for each episode):
    $S, A \leftarrow$ initial state and action of episode (e.g., $\varepsilon$-greedy)
    Repeat (for each step of episode):
        Take action $A$, observe $R, S'$
        If $S'$ is terminal:
            $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha\big[R - \hat{q}(S, A, \boldsymbol{\theta})\big]\nabla\hat{q}(S, A, \boldsymbol{\theta})$
            Go to next episode
        Choose $A'$ as a function of $\hat{q}(S', \cdot, \boldsymbol{\theta})$ (e.g., $\varepsilon$-greedy)
        $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha\big[R + \gamma\hat{q}(S', A', \boldsymbol{\theta}) - \hat{q}(S, A, \boldsymbol{\theta})\big]\nabla\hat{q}(S, A, \boldsymbol{\theta})$
        $S \leftarrow S'$
        $A \leftarrow A'$
```

# Incremental Control Algorithms

- Like prediction, we must substitute a target for $q_\pi(S, A)$
- For MC, the target is the return $G_t$

$$\Delta \mathbf{w} = \alpha \left( G_t - \hat{q}\left(S_t, A_t, \mathbf{w}\right) \right) \nabla_{\mathbf{w}} \hat{q}\left(S_t, A_t, \mathbf{w}\right)$$

- For TD(0), the target is the TD target: $R_{t+1} + \gamma Q\left(S_{t+1}, A_{t+1}\right)$

$$\Delta \mathbf{w} = \alpha \left( R_{t+1} + \gamma \hat{q}\left(S_{t+1}, A_{t+1}, \mathbf{w}\right) - \hat{q}\left(S_t, A_t, \mathbf{w}\right) \right) \nabla_{\mathbf{w}} \hat{q}\left(S_t, A_t, \mathbf{w}\right)$$

- Can we guess the deep Q learning update rule?

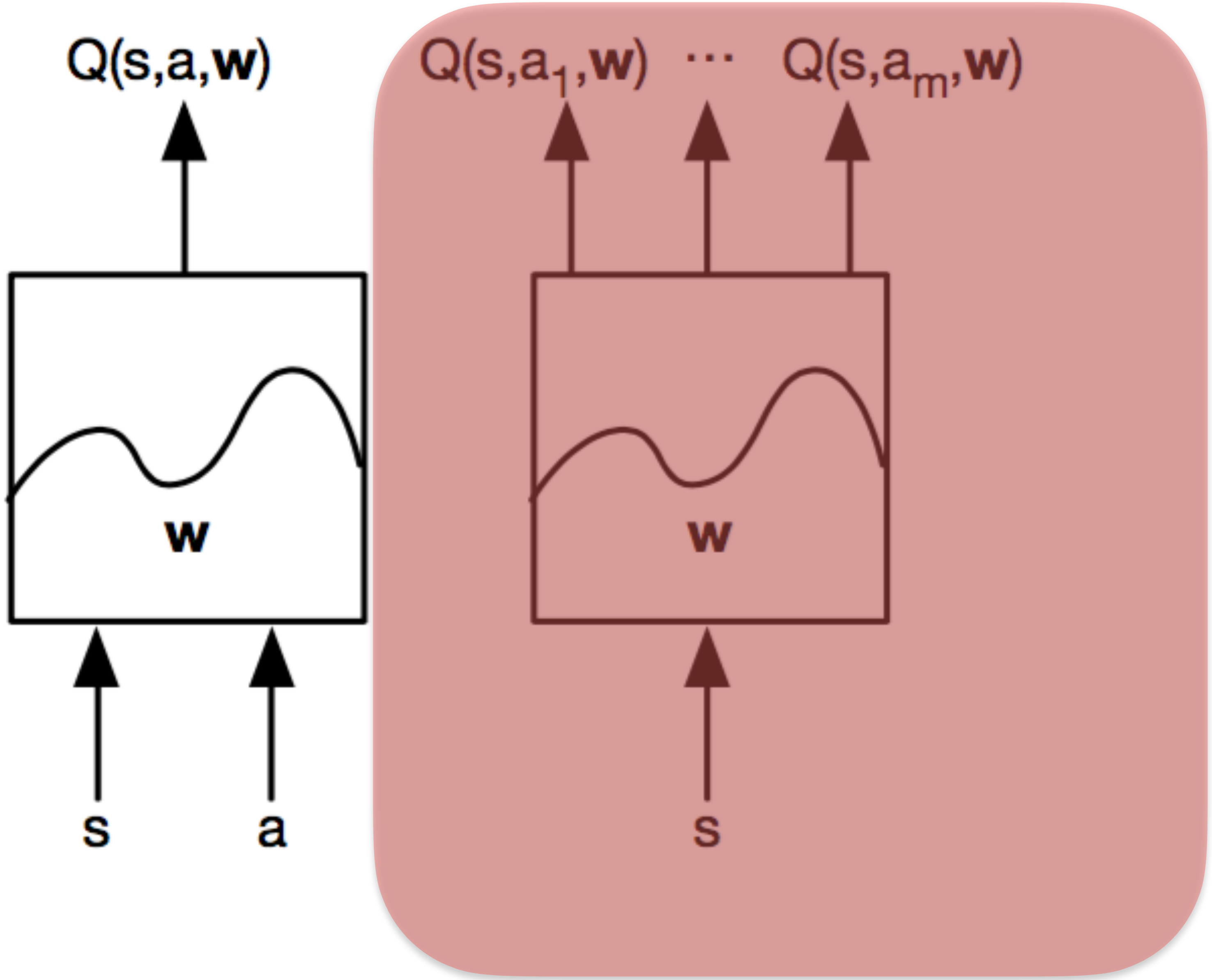
$$\Delta \mathbf{w} = \alpha(R_{t+1} + \gamma \max_{A_{t+1}} \hat{q}(S_{t+1}, A_{t+1}, \mathbf{w}) - \hat{q}(S_t, A_t, \mathbf{w}))\, \nabla_{\mathbf{w}} \hat{q}(S_t, A_t, \mathbf{w})$$

# Deep Q-Networks (DQNs)

- Represent action-state value function by Q-network with weights w

$$Q(s, a, \mathbf{w}) \approx Q^*(s, a)$$

When would this be preferred?



# Q-Learning with FA

- Minimize MSE loss by stochastic gradient descent

$$I = \left( r + \gamma \max_{a} Q(s', a', \mathbf{w}) - Q(s, a, \mathbf{w}) \right)^2$$

- Converges to Q$_*$ using table lookup representation
- But diverges using neural networks due to:
  - Correlations between samples
  - Non-stationary targets

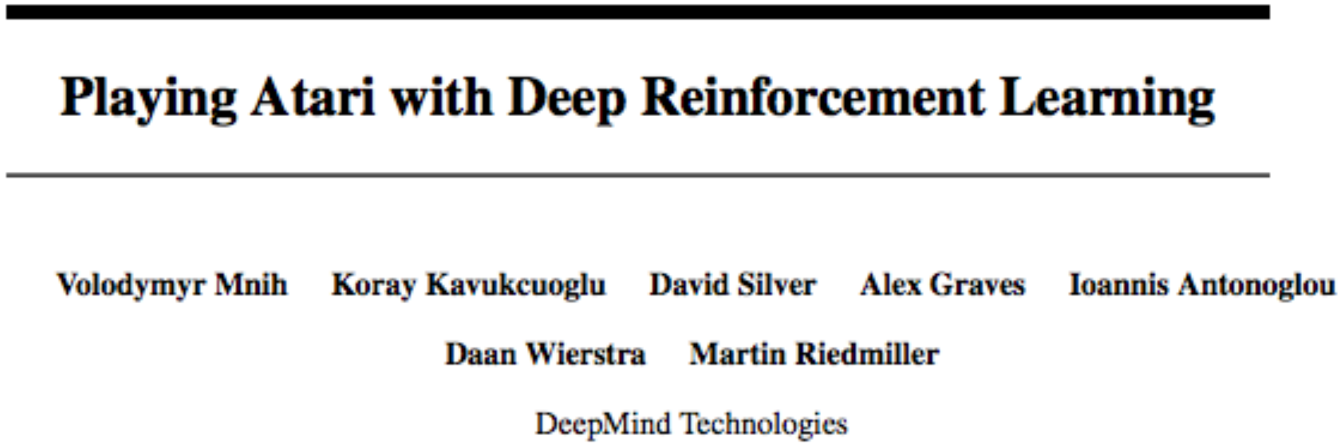
# Q-Learning

- Minimize MSE loss by stochastic gradient descent

$$I = \left( r + \gamma \max_{a} Q\left(s', a', \mathbf{w}\right) - Q(s, a, \mathbf{w}) \right)^2$$

- Converges to Q$_*$ using table lookup representation
- But diverges using neural networks due to:
  - Correlations between samples
  - Non-stationary targets

Solutions to both problems in:

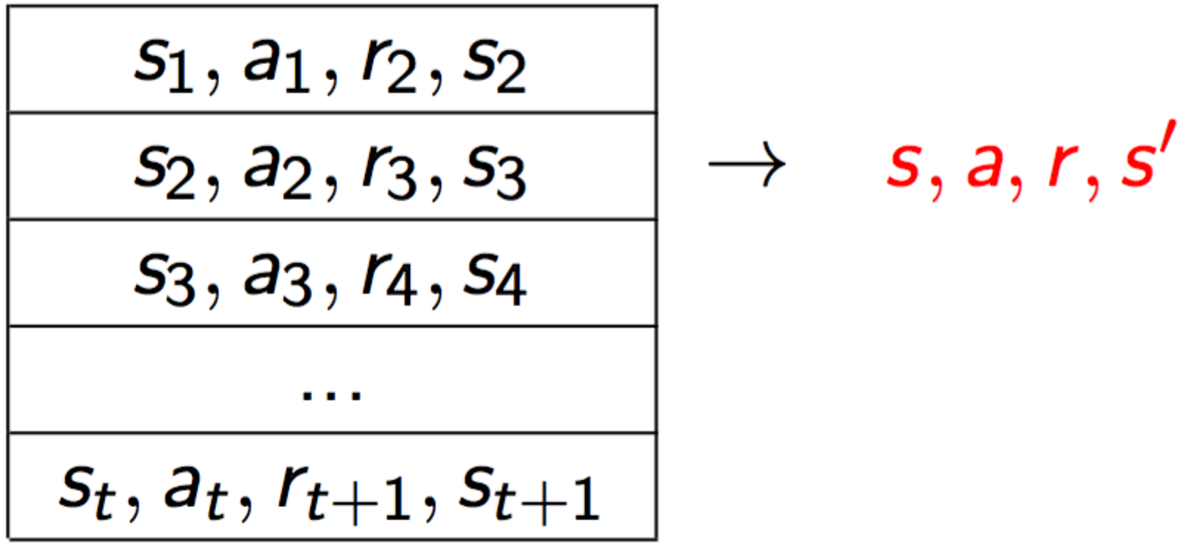


**Playing Atari with Deep Reinforcement Learning**

**Volodymyr Mnih** **Koray Kavukcuoglu** **David Silver** **Alex Graves** **Ioannis Antonoglou**
**Daan Wierstra** **Martin Riedmiller**
DeepMind Technologies

# DQN

- To remove correlations, build data-set from agent's own experience



| |
|---|
| $s_1, a_1, r_2, s_2$ |
| $s_2, a_2, r_3, s_3$ |
| $s_3, a_3, r_4, s_4$ |
| $\ldots$ |
| $s_t, a_t, r_{t+1}, s_{t+1}$ |

$\rightarrow$ $s, a, r, s'$

- Sample experiences from data-set and apply update

$$I = \left( r + \gamma \max_{a} Q\left(s', a', \mathbf{w}\right) - Q(s, a, \mathbf{w}) \right)^2$$

# DQN

- To remove correlations, build data-set from agent’s own experience

| |
|---|
| $s_1, a_1, r_2, s_2$ |
| $s_2, a_2, r_3, s_3$ |
| $s_3, a_3, r_4, s_4$ |
| $\ldots$ |
| $s_t, a_t, r_{t+1}, s_{t+1}$ |

$\rightarrow \quad s, a, r, s'$

- Sample experiences from data-set and apply update

$$I = \left( r + \gamma \max_{a} Q\left(s', a', \mathbf{w}-\right) - Q(s, a, \mathbf{w}) \right)^2$$

- To deal with non-stationarity, target parameters w− are held fixed

# Experience Replay

- Given experience consisting of ⟨state, value⟩, or <state, action/value> pairs

$$D = \left\{ \left\langle s_1, v_1^\pi \right\rangle, \left\langle s_2, v_2^\pi \right\rangle, \ldots, \left\langle s_T, v_T^\pi \right\rangle \right\}$$

- Repeat
  - Sample state, value from experience

$$\langle s, v^\pi \rangle \sim \mathcal{D}$$

  - Apply stochastic gradient descent update

$$\Delta \mathbf{w} = \alpha \left( v^\pi - \hat{v}(s, \mathbf{w}) \right) \nabla_{\mathbf{w}} \hat{v}(s, \mathbf{w})$$

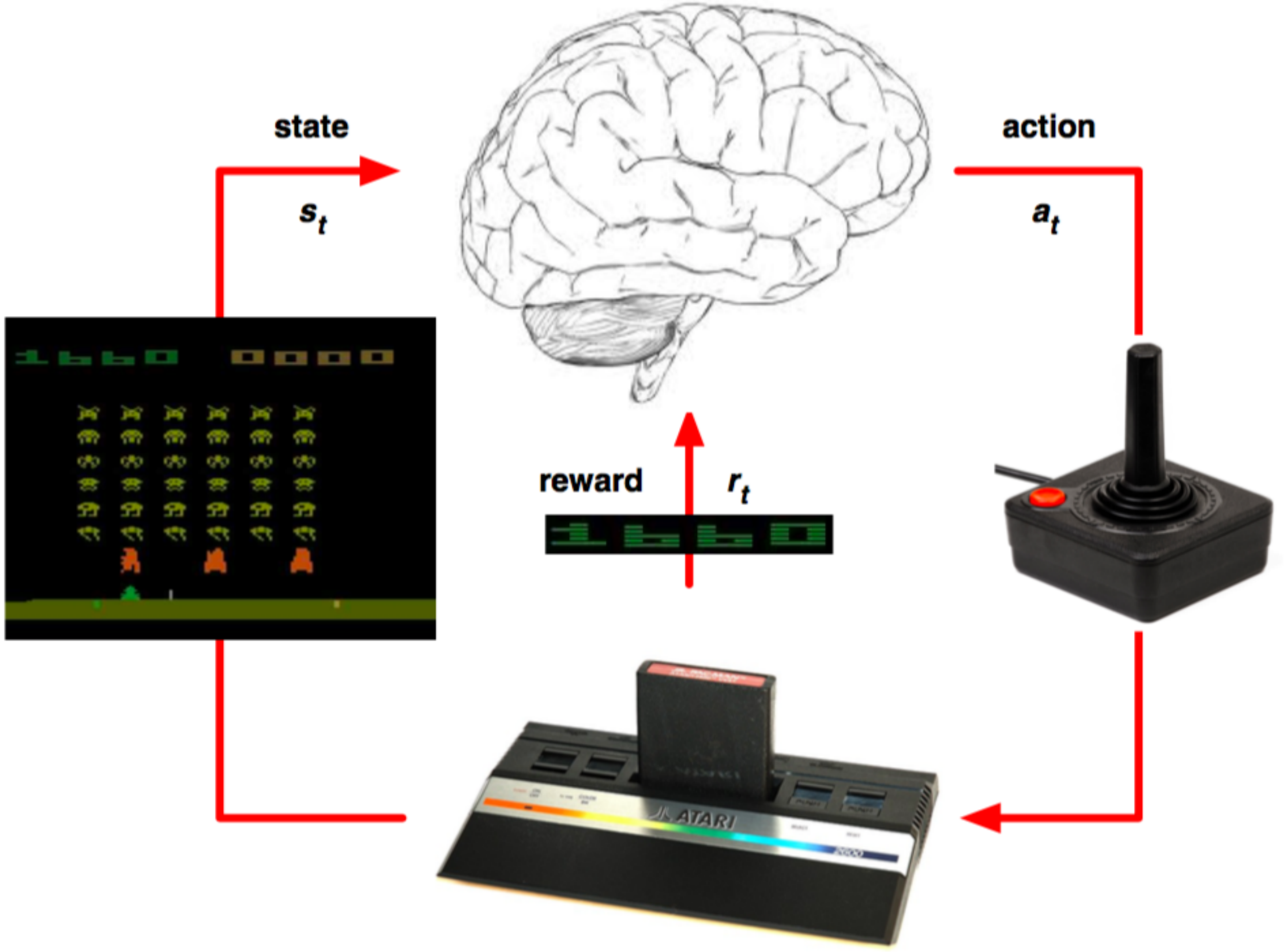
# DQNs: Experience Replay

- DQN uses experience replay and fixed Q-targets
- Store transition $(s_t, a_t, r_{t+1}, s_{t+1})$ in replay memory D
- Sample random mini-batch of transitions $(s, a, r, s')$ from D
- Compute Q-learning targets w.r.t. old, fixed parameters w−
- Optimize MSE between Q-network and Q-learning targets

$$\mathscr{L}_i\left(w_i\right) = \mathbb{E}_{s,a,r,s' \sim \mathscr{D}_i}\left[\left(\underbrace{r + \gamma \max_{a'} Q\left(s', a'; w_i^-\right)}_{\text{Q-learning target}} - \underbrace{Q\left(s, a; w_i\right)}_{\text{Q-network}}\right)^2\right]$$

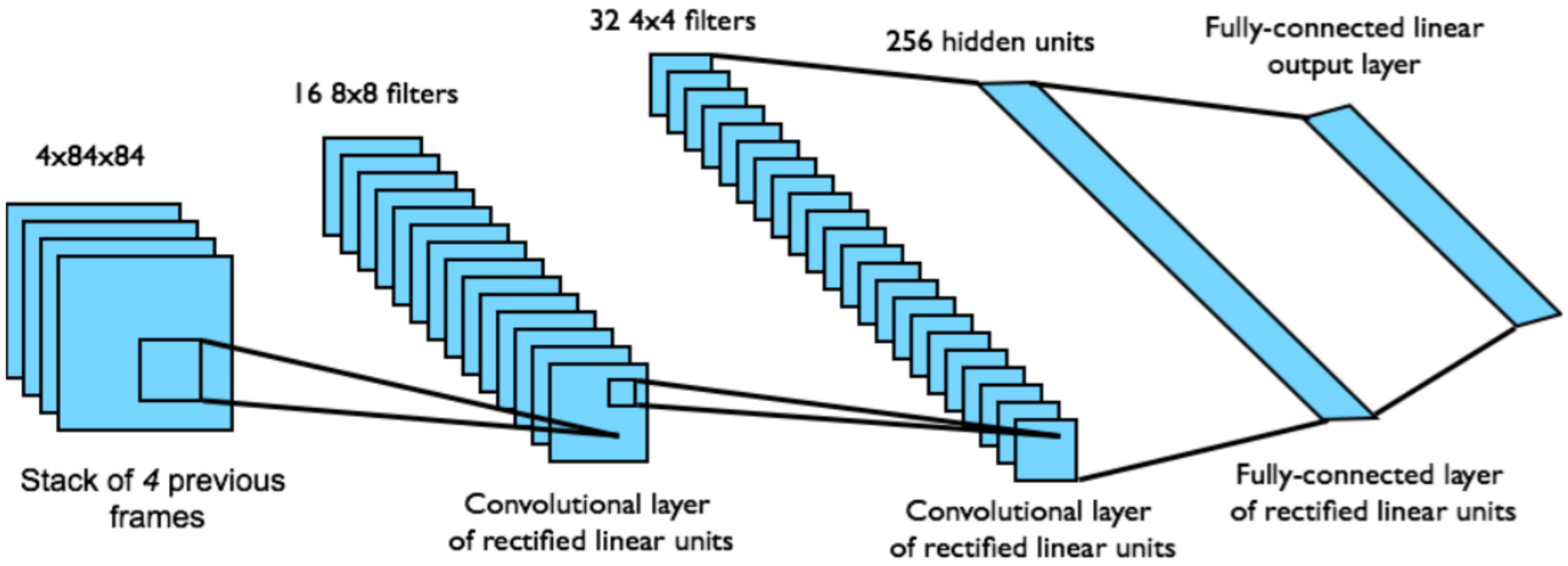
- Use stochastic gradient descent

# DQNs in Atari



# DQNs in Atari

- End-to-end learning of values $Q(s, a)$ from pixels
- Input observation is stack of raw pixels from last 4 frames
- Output is $Q(s, a)$ for 18 joystick/button positions
- Reward is change in score for that step

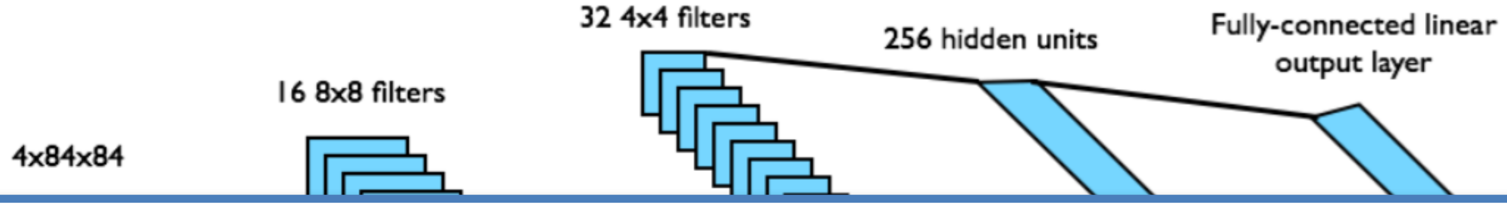


- Network architecture and hyperparameters fixed across all games

Mnih et.al., *Nature, 2014*

# DQNs in Atari

- End-to-end learning of values $Q(s, a)$ from pixels
- Input observation is stack of raw pixels from last 4 frames
- Output is $Q(s, a)$for 18 joystick/button positions
- Reward is change in score for that step



DQN source code: sites.google.com/a/deepmind.com/dqn/

- Network architecture and hyperparameters fixed across all games

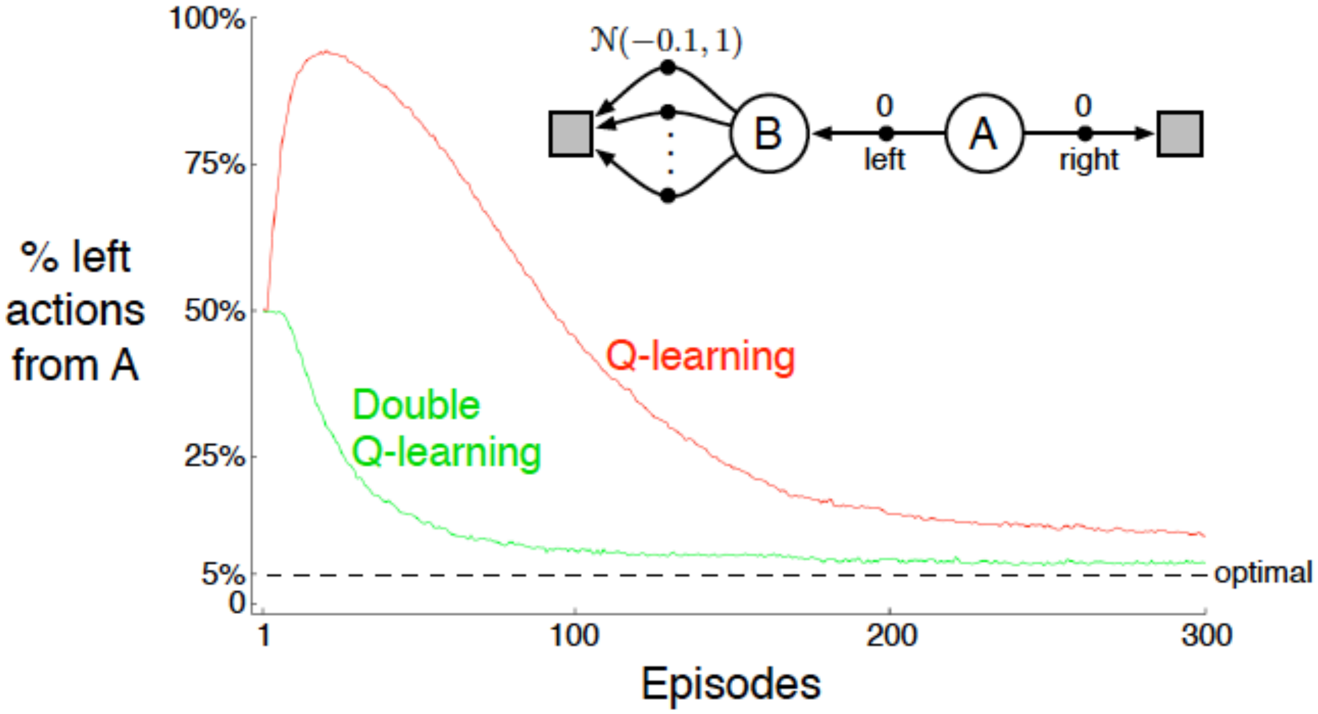
Mnih et.al., *Nature, 2014*

# Extensions

- Double Q-learning for fighting maximization bias
- Prioritized experience replay
- Multistep returns

# Maximization Bias

- We often need to maximize over our value estimates. The estimated maxima suffer from maximization bias
- Consider a state for which all ground-truth $q_*(s,a) = 0$. Our estimates $Q(s,a)$ are uncertain, some are positive and some negative.
- $Q(s, \operatorname{argmax}_a Q(s,a)) > 0$ while $q_*(s, \operatorname{argmax}_a q_*(s,a)) = 0$.
- This is because we use the same estimate Q both to choose the argmax and to evaluate it.



# Double Tabular Q-Learning

```
Initialize $Q_1(s, a)$ and $Q_2(s, a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily
Initialize $Q_1(\textit{terminal-state}, \cdot) = Q_2(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
    Initialize $S$
    Repeat (for each step of episode):
        Choose $A$ from $S$ using policy derived from $Q_1$ and $Q_2$ (e.g., $\varepsilon$-greedy in $Q_1 + Q_2$)
        Take action $A$, observe $R$, $S'$
        With 0.5 probabilility:
            $Q_1(S, A) \leftarrow Q_1(S, A) + \alpha\Big(R + \gamma Q_2\big(S', \arg\max_a Q_1(S', a)\big) - Q_1(S, A)\Big)$
        else:
            $Q_2(S, A) \leftarrow Q_2(S, A) + \alpha\Big(R + \gamma Q_1\big(S', \arg\max_a Q_2(S', a)\big) - Q_2(S, A)\Big)$
        $S \leftarrow S'$;
    until $S$ is terminal
```

Hado van Hasselt 2010

# Double Deep Q-Learning

- Current Q-network w is used to select actions
- Older Q-network w− is used to evaluate actions

$$I = \Bigg( \overbrace{r + \gamma Q\Big( s', \underbrace{\operatorname*{argmax}_{a'} Q(s', a', \mathbf{w})}_{\text{Action selection: w}}, \mathbf{w}^{-} \Big)}^{\text{Action evaluation: w−}} - Q(s, a, \mathbf{w}) \Bigg)^2$$

van Hasselt, Guez, Silver, 2015

# Prioritized Replay

- Weight experience according to “surprise” (or error)
- Store experience in priority queue according to DQN error

$$\underbrace{\left| r + \gamma \max_{a'} Q\left(s', a', \mathbf{w}^{-}\right) - Q(s, a, w) \right|}$$

$p_i$ is proportional to DQN error

- Stochastic Prioritization

$$P(i) = \frac{p_i^{\alpha}}{\sum_k p_k^{\alpha}}$$

- $\alpha$ determines how much prioritization is used, with $\alpha = 0$ corresponding to the uniform case.

*Schaul, Quan, Antonoglou, Silver, ICLR 2016*

# Multistep Returns

- Truncated n-step return from a state s_t:

$$R_t^{(n)} = \sum_{k=0}^{n-1} \gamma_t^{(k)} R_{t+k+1}$$

- Multistep Q-learning update rule:

$$I = \left(R_t^{(n)} + \gamma_t^{(n)} \max\nolimits_{a'} Q(S_{t+n}, a', \mathbf{w}) - Q(s, a, \mathbf{w})\right)^2$$

- Single step Q-learning update rule:

$$I = \left(r + \gamma \max_{a} Q(s', a', \mathbf{w}) - Q(s, a, \mathbf{w})\right)^2$$

# n-step TD Returns/Targets

- Monte Carlo: $G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots + \gamma^{T-t-1} R_T$

# n-step TD Returns/Targets

- Monte Carlo: $G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots + \gamma^{T-t-1} R_T$
- TD: $G_t^{(1)} \doteq R_{t+1} + \gamma V_t(S_{t+1})$
  - Use V_t to estimate remaining return

# n-step TD Returns/Targets

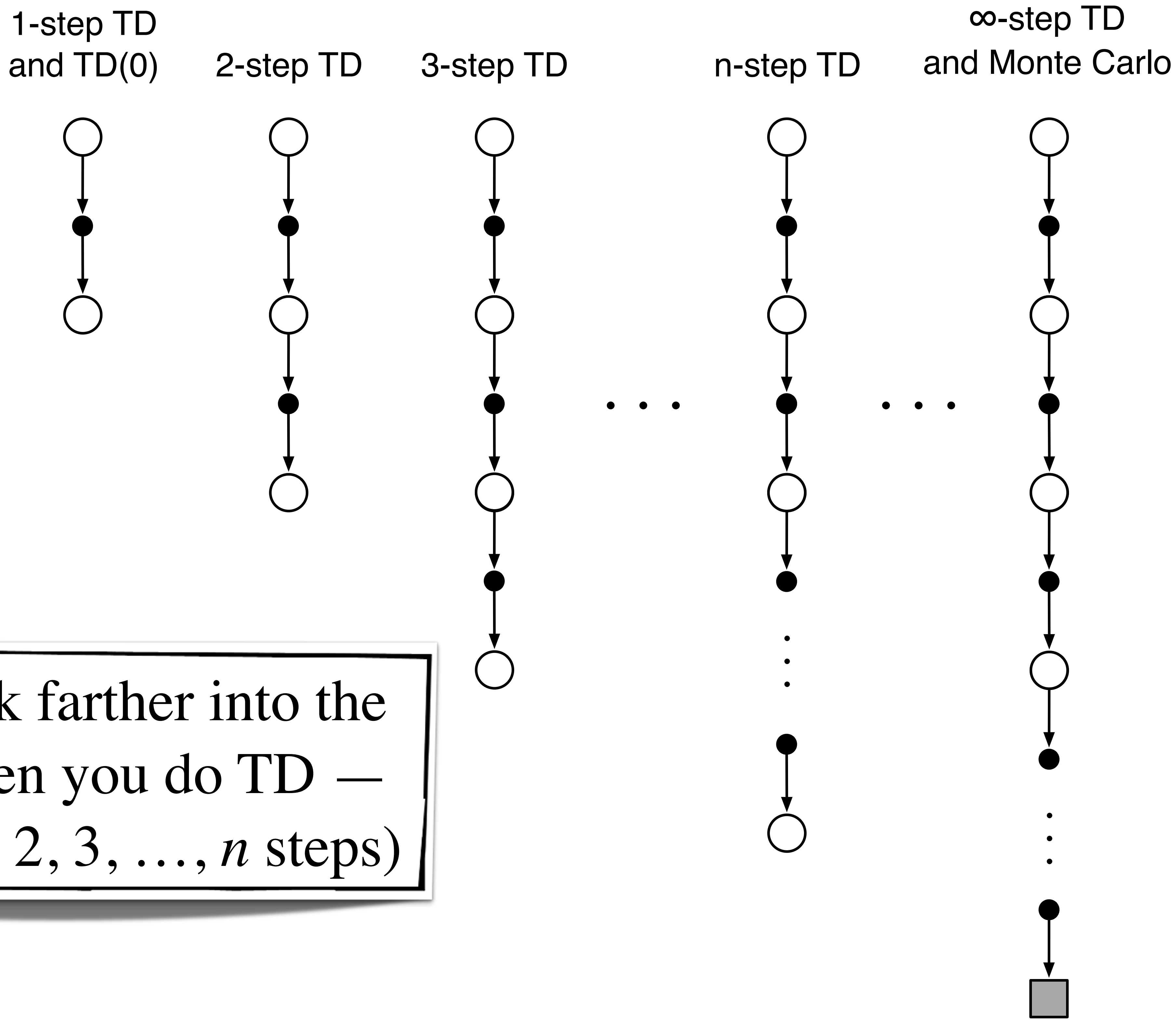
- Monte Carlo: $G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots + \gamma^{T-t-1} R_T$
- TD: $G_t^{(1)} \doteq R_{t+1} + \gamma V_t(S_{t+1})$
  - Use Vt to estimate remaining return
- n-step TD:
  - 2 step return: $G_t^{(2)} \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 V_t(S_{t+2})$

# n-step TD Returns/Targets

- Monte Carlo: $G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots + \gamma^{T-t-1} R_T$
- TD: $G_t^{(1)} \doteq R_{t+1} + \gamma V_t(S_{t+1})$
  - Use Vt to estimate remaining return
- n-step TD:
  - 2 step return: $G_t^{(2)} \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 V_t(S_{t+2})$
  - n-step return: $G_t^{(n)} \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n V_t(S_{t+n})$

with $G_t^{(n)} \doteq G_t \text{ if } t + n \geq T$

# n-step TD Prediction



Idea: Look farther into the future when you do TD — backup $(1, 2, 3, \ldots, n$ steps)

# *n*-step TD

- Recall the n-step return:

$$G_t^{(n)} \doteq R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n V_{t+n-1}(S_{t+n}), \quad n \geq 1, 0 \leq t < T - n$$

- Of course, this is not available until time $t + n$

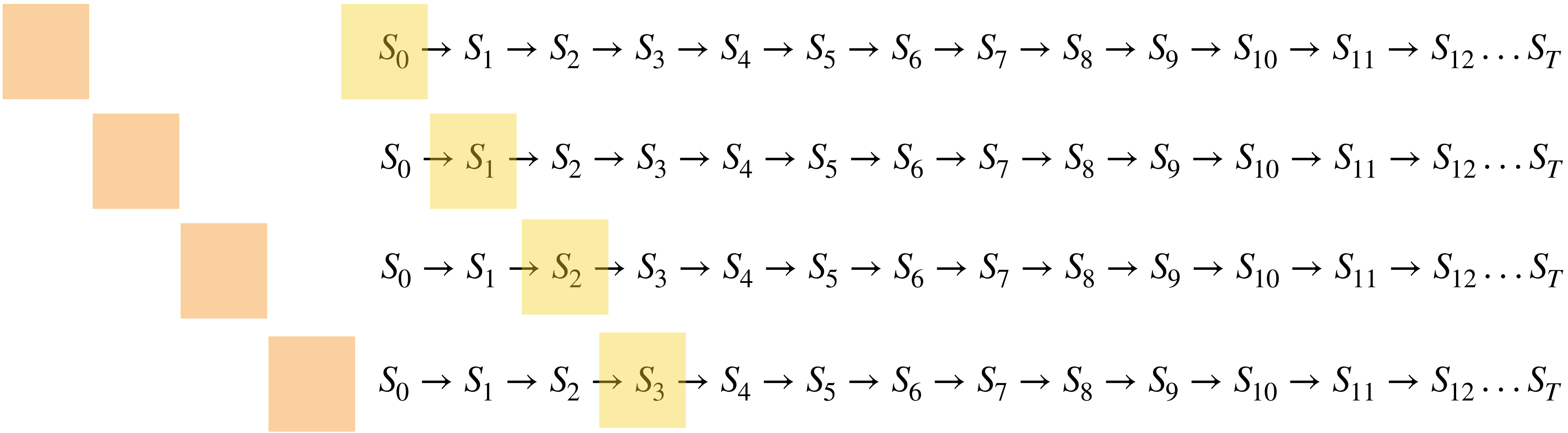
- The natural algorithm is thus to wait until then:

$$V_{t+n}(S_t) \doteq V_{t+n-1}(S_t) + \alpha \left[ G_t^{(n)} - V_{t+n-1}(S_t) \right], \qquad 0 \leq t < T,$$

- This is called $n$-step TD

**$n$-step TD for estimating $V \approx v_\pi$**

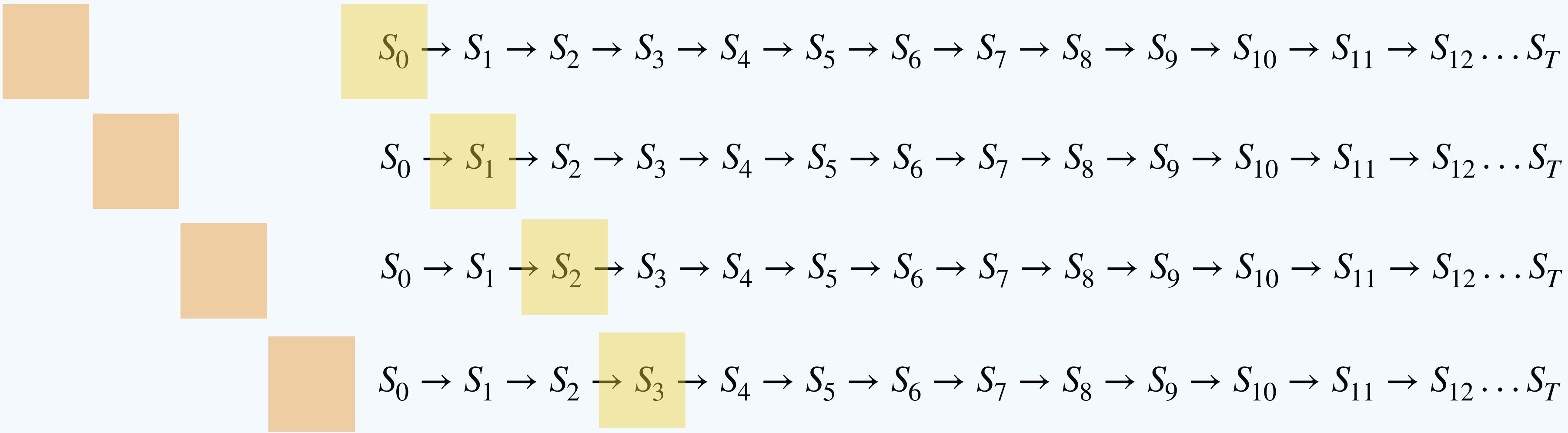
```
Initialize $V(s)$ arbitrarily, $s \in \mathcal{S}$
Parameters: step size $\alpha \in (0, 1]$, a positive integer $n$
All store and access operations (for $S_t$ and $R_t$) can take their index mod $n$

Repeat (for each episode):
    Initialize and store $S_0 \neq$ terminal
    $T \leftarrow \infty$
    For $t = 0, 1, 2, \ldots$ :
    |   If $t < T$, then:
    |       Take an action according to $\pi(\cdot|S_t)$
    |       Observe and store the next reward as $R_{t+1}$ and the next state as $S_{t+1}$
    |       If $S_{t+1}$ is terminal, then $T \leftarrow t + 1$
    |   $\tau \leftarrow t - n + 1$    ($\tau$ is the time whose state's estimate is being updated)
    |   If $\tau \geq 0$:
    |       $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
    |       If $\tau + n < T$, then: $G \leftarrow G + \gamma^n V(S_{\tau+n})$                ($G_\tau^{(n)}$)
    |       $V(S_\tau) \leftarrow V(S_\tau) + \alpha\,[G - V(S_\tau)]$
    Until $\tau = T - 1$
```



**$n$-step TD for estimating $V \approx v_\pi$**

```
Initialize $V(s)$ arbitrarily, $s \in \mathcal{S}$
Parameters: step size $\alpha \in (0,1]$, a positive integer $n$
All store and access operations (for $S_t$ and $R_t$) can take their index mod $n$

Repeat (for each episode):
   Initialize and store $S_0 \neq$ terminal
   $T \leftarrow \infty$
   For $t = 0, 1, 2, \ldots$ :
   |  If $t < T$, then:
   |      Take an action according to $\pi(\cdot|S_t)$
   |      Observe and store the next reward as $R_{t+1}$ and the next state as $S_{t+1}$
   |      If $S_{t+1}$ is terminal, then $T \leftarrow t+1$
   |  $\tau \leftarrow t - n + 1$    ($\tau$ is the time whose state's estimate is being updated)
   |  If $\tau \geq 0$:
   |      $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
   |      If $\tau + n < T$, then: $G \leftarrow G + \gamma^n V(S_{\tau+n})$          $(G_\tau^{(n)})$
   |      $V(S_\tau) \leftarrow V(S_\tau) + \alpha\,[G - V(S_\tau)]$
   Until $\tau = T - 1$
```

No value update



$S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow S_5 \rightarrow S_6 \rightarrow S_7 \rightarrow S_8 \rightarrow S_9 \rightarrow S_{10} \rightarrow S_{11} \rightarrow S_{12} \ldots S_T$

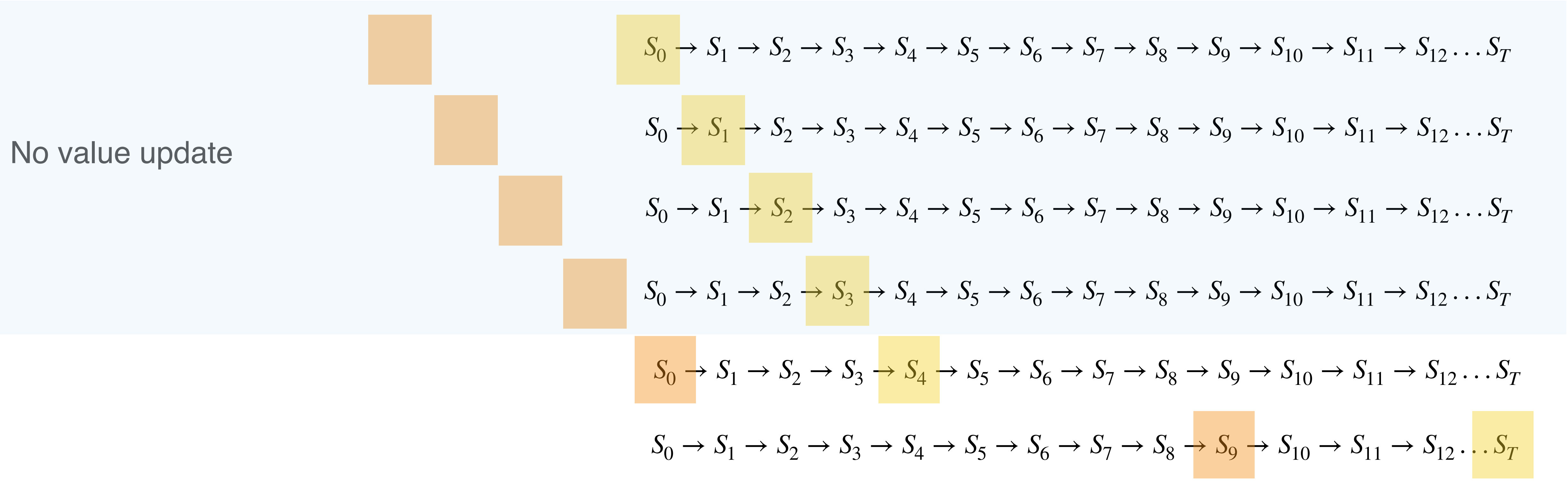
$S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow S_5 \rightarrow S_6 \rightarrow S_7 \rightarrow S_8 \rightarrow S_9 \rightarrow S_{10} \rightarrow S_{11} \rightarrow S_{12} \ldots S_T$

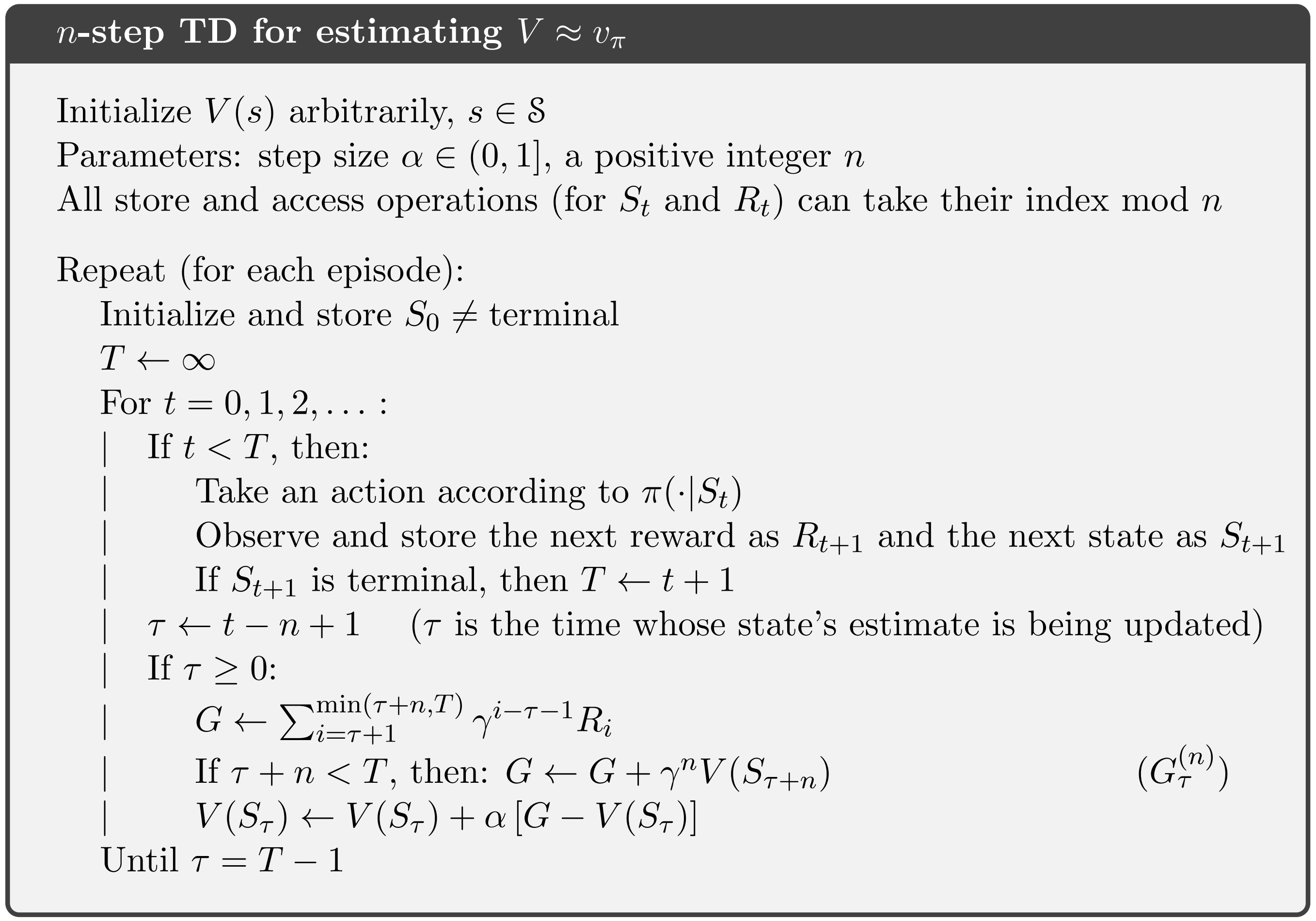
$S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow S_5 \rightarrow S_6 \rightarrow S_7 \rightarrow S_8 \rightarrow S_9 \rightarrow S_{10} \rightarrow S_{11} \rightarrow S_{12} \ldots S_T$

$S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow S_5 \rightarrow S_6 \rightarrow S_7 \rightarrow S_8 \rightarrow S_9 \rightarrow S_{10} \rightarrow S_{11} \rightarrow S_{12} \ldots S_T$

## $n$-step TD for estimating $V \approx v_\pi$

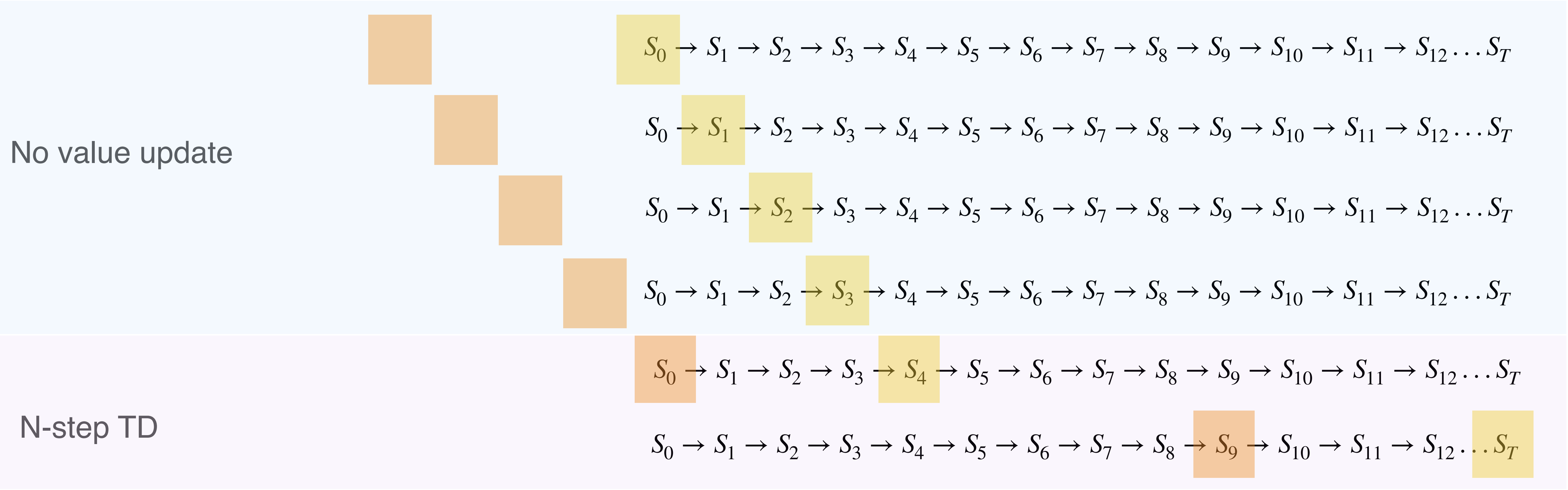
```
Initialize $V(s)$ arbitrarily, $s \in \mathcal{S}$
Parameters: step size $\alpha \in (0,1]$, a positive integer $n$
All store and access operations (for $S_t$ and $R_t$) can take their index mod $n$

Repeat (for each episode):
    Initialize and store $S_0 \neq$ terminal
    $T \leftarrow \infty$
    For $t = 0, 1, 2, \ldots$ :
    |   If $t < T$, then:
    |       Take an action according to $\pi(\cdot|S_t)$
    |       Observe and store the next reward as $R_{t+1}$ and the next state as $S_{t+1}$
    |       If $S_{t+1}$ is terminal, then $T \leftarrow t+1$
    |   $\tau \leftarrow t - n + 1$    ($\tau$ is the time whose state's estimate is being updated)
    |   If $\tau \geq 0$:
    |       $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
    |       If $\tau + n < T$, then: $G \leftarrow G + \gamma^n V(S_{\tau+n})$                ($G_\tau^{(n)}$)
    |       $V(S_\tau) \leftarrow V(S_\tau) + \alpha\,[G - V(S_\tau)]$
    Until $\tau = T - 1$
```





**$n$-step TD for estimating $V \approx v_\pi$**

```
Initialize $V(s)$ arbitrarily, $s \in \mathcal{S}$
Parameters: step size $\alpha \in (0,1]$, a positive integer $n$
All store and access operations (for $S_t$ and $R_t$) can take their index mod $n$

Repeat (for each episode):
    Initialize and store $S_0 \neq$ terminal
    $T \leftarrow \infty$
    For $t = 0, 1, 2, \ldots$ :
    |   If $t < T$, then:
    |       Take an action according to $\pi(\cdot|S_t)$
    |       Observe and store the next reward as $R_{t+1}$ and the next state as $S_{t+1}$
    |       If $S_{t+1}$ is terminal, then $T \leftarrow t+1$
    |   $\tau \leftarrow t - n + 1$    ($\tau$ is the time whose state's estimate is being updated)
    |   If $\tau \geq 0$:
    |       $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
    |       If $\tau + n < T$, then: $G \leftarrow G + \gamma^n V(S_{\tau+n})$                  $(G_\tau^{(n)})$
    |       $V(S_\tau) \leftarrow V(S_\tau) + \alpha\,[G - V(S_\tau)]$
    Until $\tau = T - 1$
```



No value update

$S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow S_5 \rightarrow S_6 \rightarrow S_7 \rightarrow S_8 \rightarrow S_9 \rightarrow S_{10} \rightarrow S_{11} \rightarrow S_{12} \ldots S_T$

$S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow S_5 \rightarrow S_6 \rightarrow S_7 \rightarrow S_8 \rightarrow S_9 \rightarrow S_{10} \rightarrow S_{11} \rightarrow S_{12} \ldots S_T$

$S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow S_5 \rightarrow S_6 \rightarrow S_7 \rightarrow S_8 \rightarrow S_9 \rightarrow S_{10} \rightarrow S_{11} \rightarrow S_{12} \ldots S_T$

$S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow S_5 \rightarrow S_6 \rightarrow S_7 \rightarrow S_8 \rightarrow S_9 \rightarrow S_{10} \rightarrow S_{11} \rightarrow S_{12} \ldots S_T$

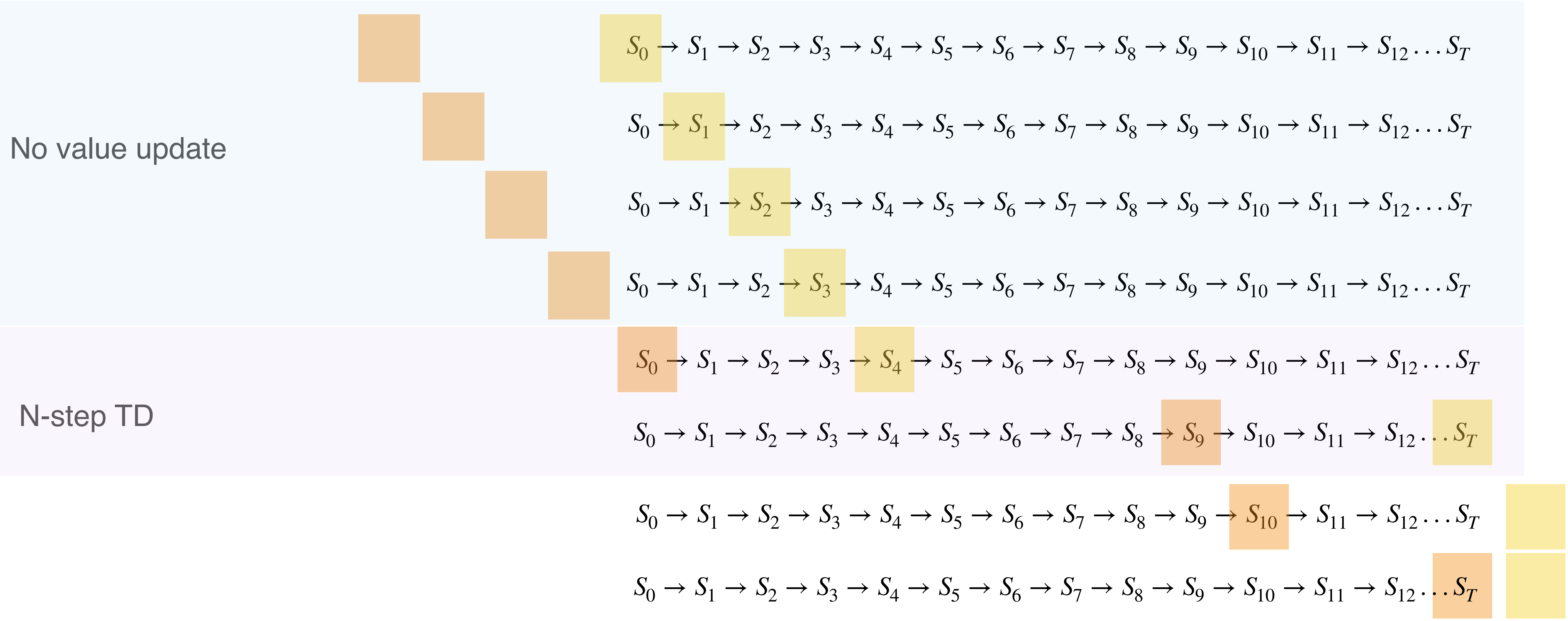
N-step TD

$S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow S_5 \rightarrow S_6 \rightarrow S_7 \rightarrow S_8 \rightarrow S_9 \rightarrow S_{10} \rightarrow S_{11} \rightarrow S_{12} \ldots S_T$

$S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow S_5 \rightarrow S_6 \rightarrow S_7 \rightarrow S_8 \rightarrow S_9 \rightarrow S_{10} \rightarrow S_{11} \rightarrow S_{12} \ldots S_T$

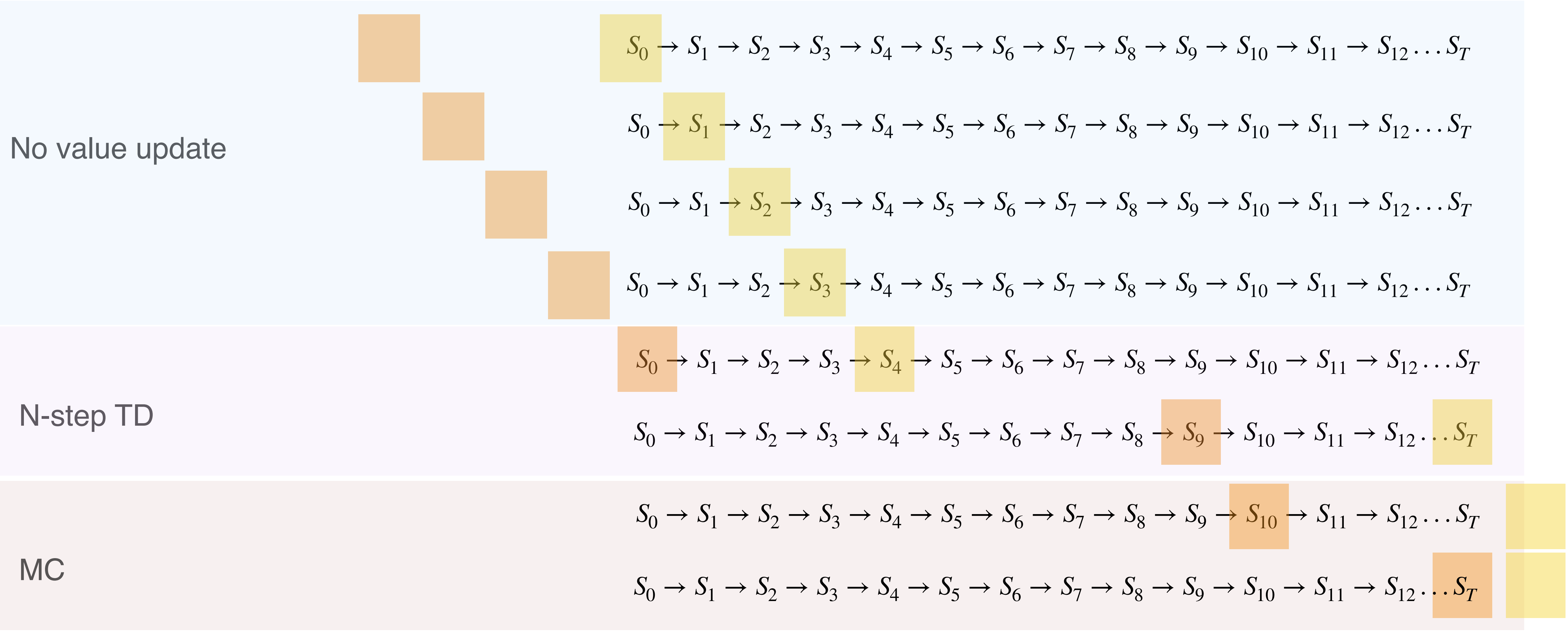
**$n$-step TD for estimating $V \approx v_\pi$**

```
Initialize $V(s)$ arbitrarily, $s \in \mathcal{S}$
Parameters: step size $\alpha \in (0,1]$, a positive integer $n$
All store and access operations (for $S_t$ and $R_t$) can take their index mod $n$

Repeat (for each episode):
    Initialize and store $S_0 \neq$ terminal
    $T \leftarrow \infty$
    For $t = 0, 1, 2, \ldots$ :
    |   If $t < T$, then:
    |       Take an action according to $\pi(\cdot|S_t)$
    |       Observe and store the next reward as $R_{t+1}$ and the next state as $S_{t+1}$
    |       If $S_{t+1}$ is terminal, then $T \leftarrow t+1$
    |   $\tau \leftarrow t - n + 1$   ($\tau$ is the time whose state's estimate is being updated)
    |   If $\tau \geq 0$:
    |       $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
    |       If $\tau + n < T$, then: $G \leftarrow G + \gamma^n V(S_{\tau+n})$            $(G_\tau^{(n)})$
    |       $V(S_\tau) \leftarrow V(S_\tau) + \alpha\,[G - V(S_\tau)]$
    Until $\tau = T - 1$
```

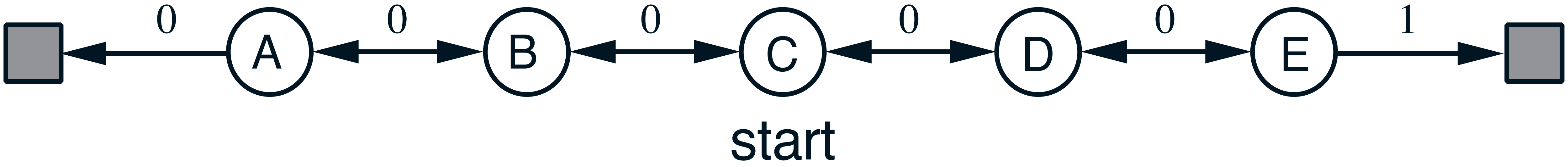


**$n$-step TD for estimating $V \approx v_\pi$**

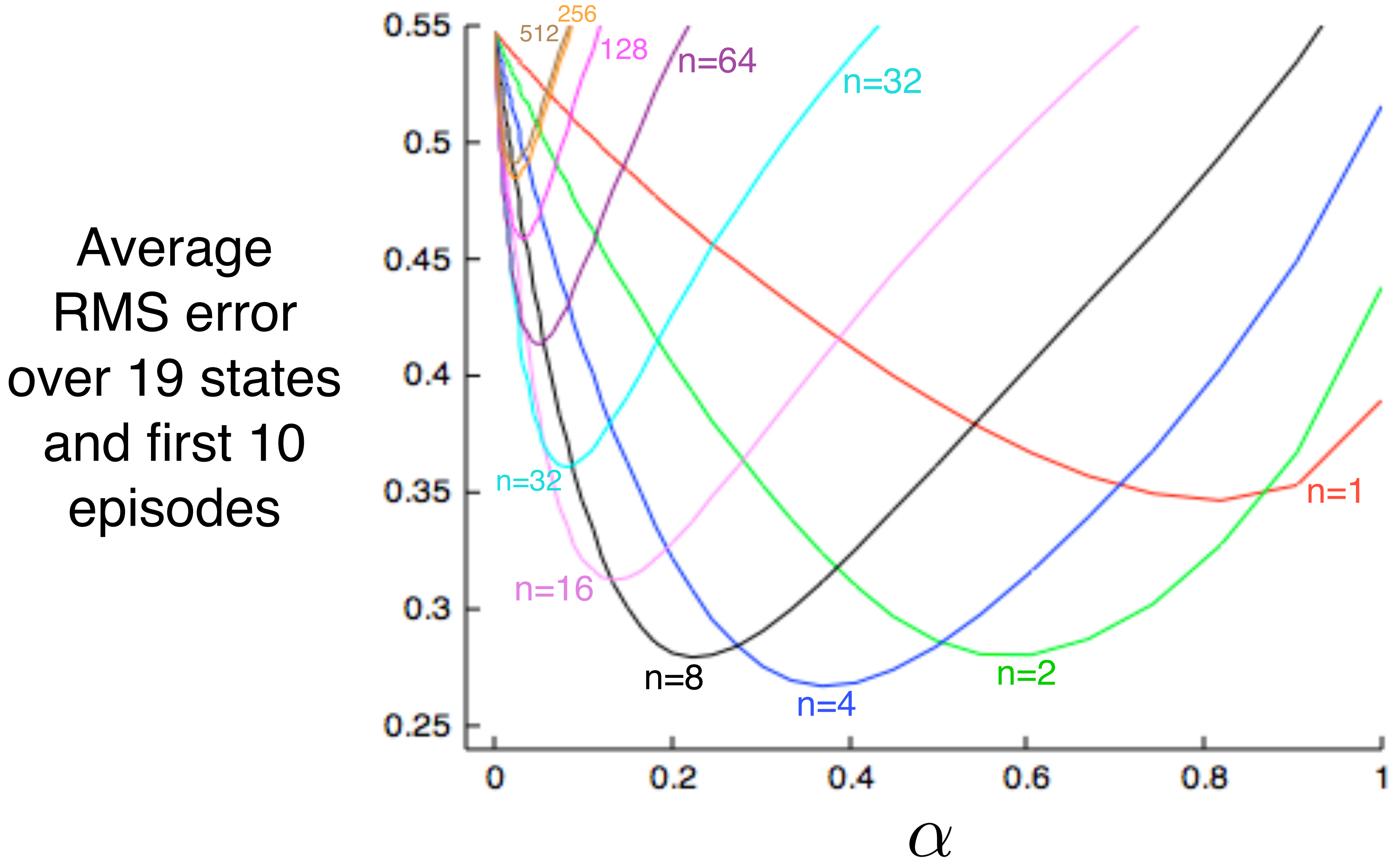
```
Initialize $V(s)$ arbitrarily, $s \in \mathcal{S}$
Parameters: step size $\alpha \in (0,1]$, a positive integer $n$
All store and access operations (for $S_t$ and $R_t$) can take their index mod $n$

Repeat (for each episode):
    Initialize and store $S_0 \neq$ terminal
    $T \leftarrow \infty$
    For $t = 0, 1, 2, \ldots$ :
    |   If $t < T$, then:
    |       Take an action according to $\pi(\cdot|S_t)$
    |       Observe and store the next reward as $R_{t+1}$ and the next state as $S_{t+1}$
    |       If $S_{t+1}$ is terminal, then $T \leftarrow t+1$
    |   $\tau \leftarrow t - n + 1$   ($\tau$ is the time whose state's estimate is being updated)
    |   If $\tau \geq 0$:
    |       $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
    |       If $\tau + n < T$, then: $G \leftarrow G + \gamma^n V(S_{\tau+n})$          $(G_\tau^{(n)})$
    |       $V(S_\tau) \leftarrow V(S_\tau) + \alpha [G - V(S_\tau)]$
    Until $\tau = T - 1$
```



# Random Walk Examples



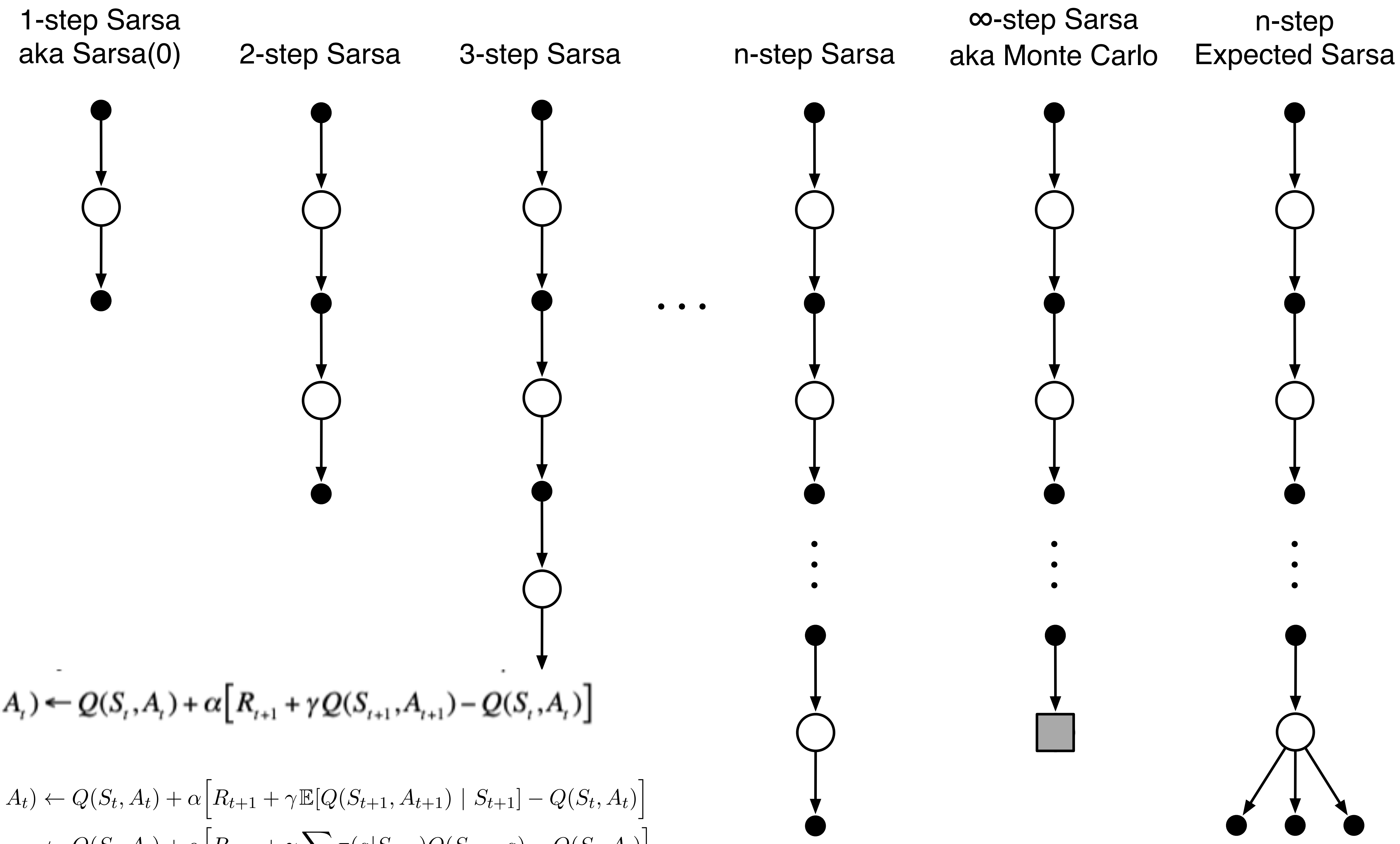
# A Larger Example – 19-state Random Walk



n-step TD results

- An intermediate $\alpha$ is best
- An intermediate $n$ is best

# It's much the same for action values



1-step Sarsa
aka Sarsa(0)

2-step Sarsa

3-step Sarsa

n-step Sarsa

∞-step Sarsa
aka Monte Carlo

n-step
Expected Sarsa

$$Q(S_t,A_t) \leftarrow Q(S_t,A_t) + \alpha\big[R_{t+1} + \gamma Q(S_{t+1},A_{t+1}) - Q(S_t,A_t)\big]$$

$$\begin{aligned} Q(S_t, A_t) &\leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma \mathbb{E}[Q(S_{t+1}, A_{t+1}) \mid S_{t+1}] - Q(S_t, A_t)\Big] \\ &\leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma \sum \pi(a|S_{t+1}) Q(S_{t+1}, a) - Q(S_t, A_t)\Big] \end{aligned}$$

# On-policy *n*-step Action-value Methods

- Action-value form of n-step return

$$G_t^{(n)} \doteq R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n \underline{Q_{t+n-1}(S_{t+n}, A_{t+n})}$$

- $n$-step Sarsa:

$$Q_{t+n}(S_t, A_t) \doteq Q_{t+n-1}(S_t, A_t) + \alpha \left[ G_t^{(n)} - Q_{t+n-1}(S_t, A_t) \right]$$

- $n$-step Expected Sarsa is the same update with a slightly different n-step return:

$$G_t^{(n)} \doteq R_{t+1} + \cdots + \gamma^{n-1} R_{t+n} + \underline{\gamma^n \sum_a \pi(a|S_{t+n}) Q_{t+n-1}(S_{t+n}, a)}$$

# Off-policy *n*-step Methods by Importance Sampling

- Recall the *importance-sampling ratio*:

$$\rho_t^{t+n} \doteq \prod_{k=t}^{\min(t+n-1,T-1)} \frac{\pi(A_k|S_k)}{\mu(A_k|S_k)}$$

- We get off-policy methods by weighting updates by this ratio

- Off-policy $n$-step TD:

$$V_{t+n}(S_t) \doteq V_{t+n-1}(S_t) + \underline{\alpha\rho_t^{t+n}} \left[G_t^{(n)} - V_{t+n-1}(S_t)\right]$$

- Off-policy $n$-step Sarsa:

$$Q_{t+n}(S_t, A_t) \doteq Q_{t+n-1}(S_t, A_t) + \alpha\underline{\rho_{t+1}^{t+n}} \left[G_t^{(n)} - Q_{t+n-1}(S_t, A_t)\right]$$

- Off-policy $n$-step Expected Sarsa:

$$Q_{t+n}(S_t, A_t) \doteq Q_{t+n-1}(S_t, A_t) + \alpha\, \rho_{t+1}^{\underline{t+n-1}} \left[G_t^{(n)} - Q_{t+n-1}(S_t, A_t)\right]$$

# Conclusions Regarding *n*-step Methods

- Generalize Temporal-Difference and Monte Carlo learning methods, sliding from one to the other as n increases
  - $n = 1$ is TD as in Chapter 6
  - $n = \infty$ is MC as in Chapter 5
  - an intermediate $n$ is often much better than either extreme
  - applicable to both continuing and episodic problems
- There is some cost in computation
  - need to remember the last $n$ states
  - learning is delayed by $n$ steps
  - per-step computation is small and uniform, like TD