Carnegie Mellon
School of Computer Science

Deep Reinforcement Learning and Control

REINFORCE, Actor-critic methods

Fall 2021, CMU 10-703

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Value-Based and Policy-Based RL

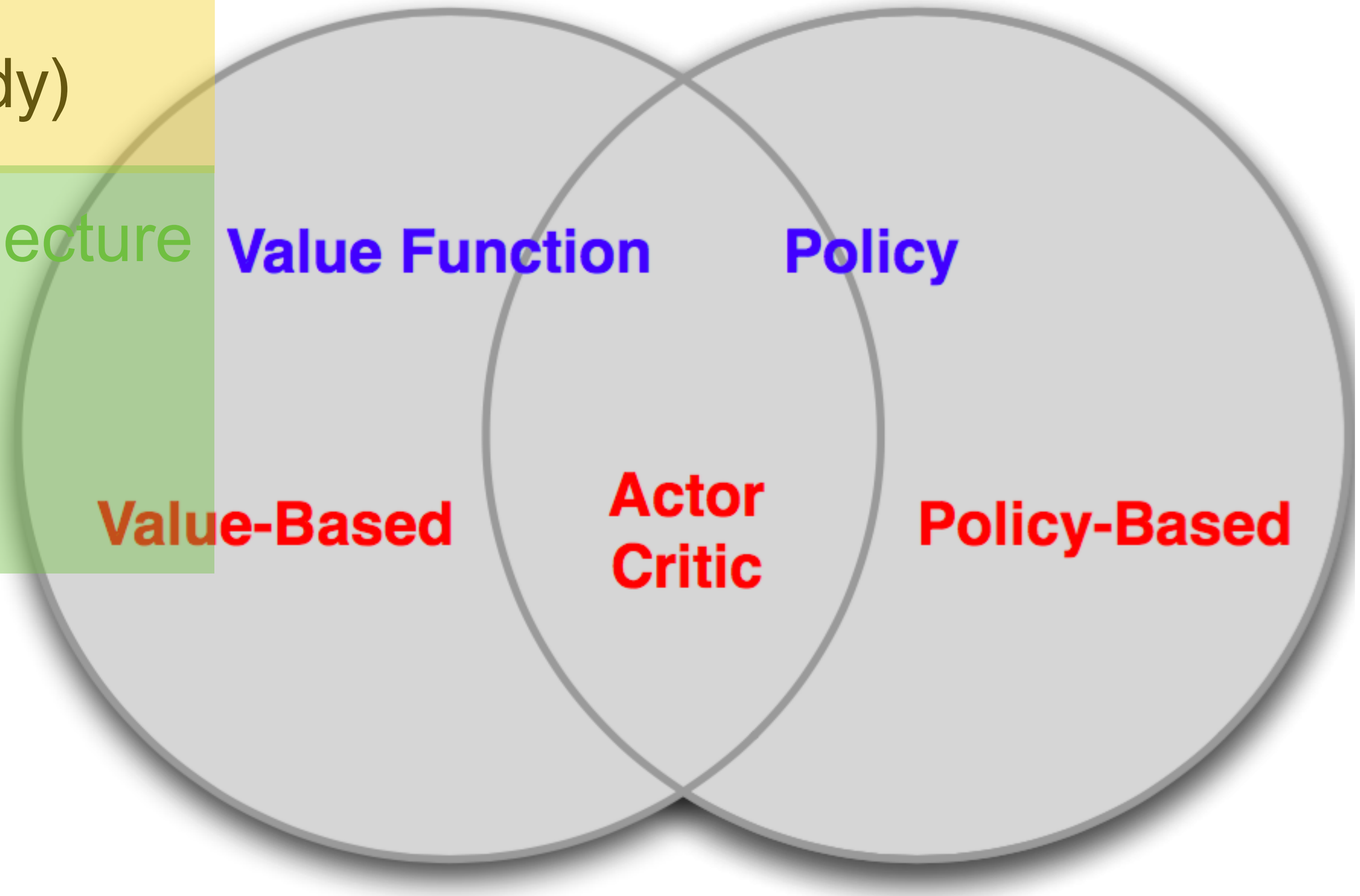
- Value Based
  - Learned Value Function
  - Implicit policy (e.g. ε-greedy)

We have covered those

- Policy Based
  - No Value Function
  - Learned Policy

this lecture

- Actor-Critic
  - Learned Value Function
  - Learned Policy



Policy Optimization

- Let $U(\theta)$ be any policy objective function

0. Initialize policy parameters θ
1. Sample trajectories $\{\tau_i = \{s_t^i, a_t^i\}_{t=0}^T\}$ by deploying the current policy $\pi_\theta(a_t | s_t)$.
2. Compute gradient vector $\nabla_\theta U(\theta)$
3. $\theta \leftarrow \theta + \alpha \nabla_\theta U(\theta)$

- Policy based reinforcement learning is an optimization problem: Find θ that maximizes $U(\theta)$

- General alternatives: gradient free optimization
  - Hill climbing
  - Genetic algorithms. Today we will learn different policy gradient estimators.

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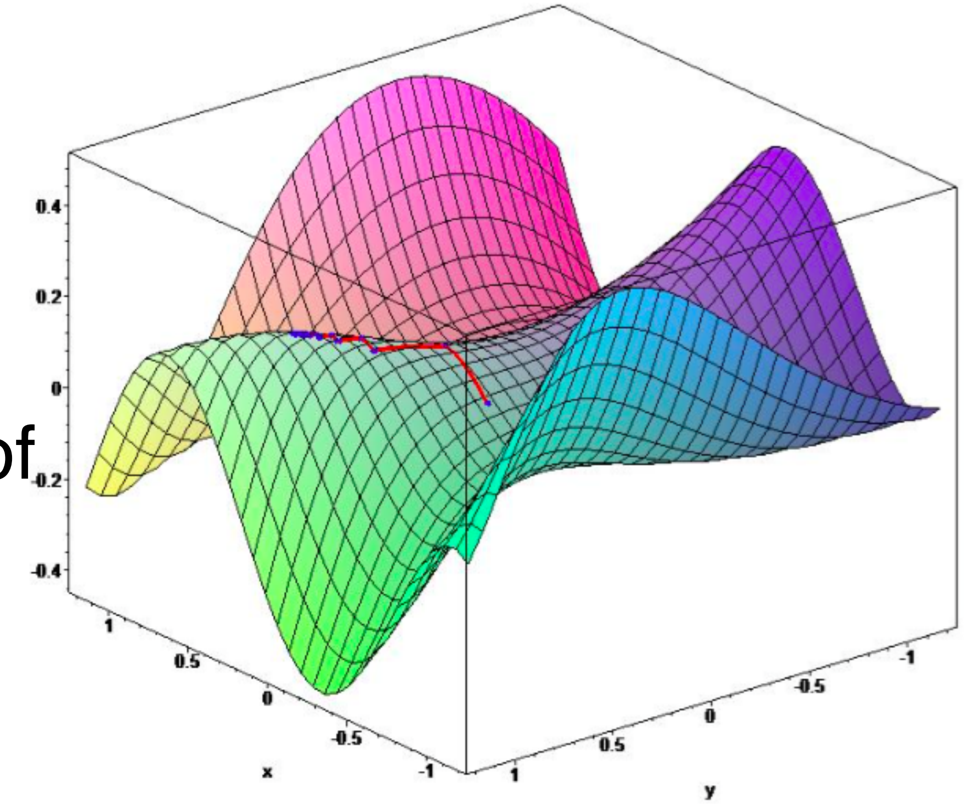
- Advantages over value based methods:
  - Effective in high-dimensional or continuous action spaces
  - Can learn stochastic policies

Policy Gradient



- Let U(θ) be any policy objective function
- Policy gradient algorithms search for a local maximum in U(θ) by ascending the gradient of the policy, w.r.t. parameters θ

$$\theta_{new} = \theta_{old} + \Delta\theta$$

$$\Delta\theta = \alpha \nabla_\theta U(\theta)$$

α is a step-size parameter (learning rate)

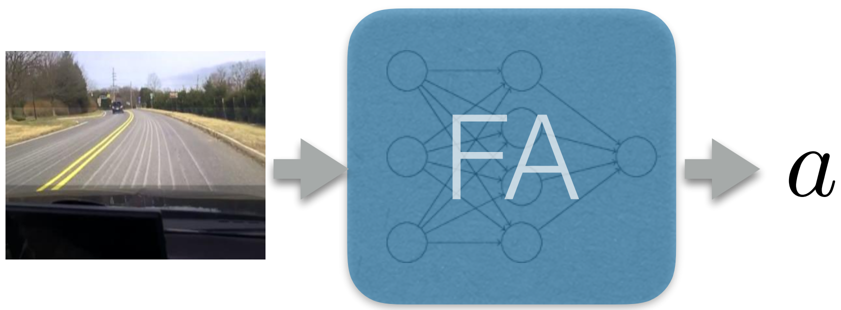
is the policy gradient

$$\nabla_\theta U(\theta) = \begin{pmatrix} \frac{\partial U(\theta)}{\partial \theta_1} \\ \vdots \\ \frac{\partial U(\theta)}{\partial \theta_n} \end{pmatrix}$$

Policy gradient: the gradient of the policy objective w.r.t. the parameters of the policy

Policy functions

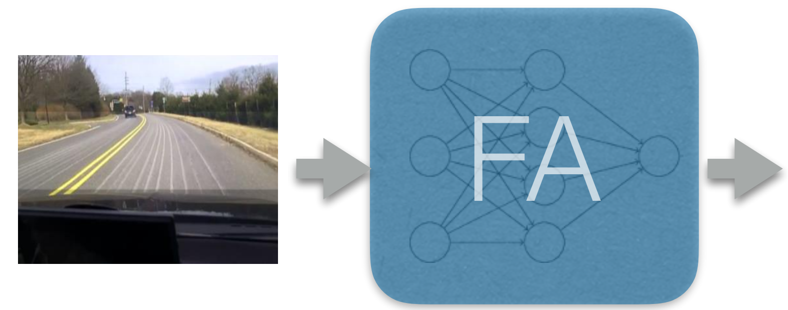
deterministic continuous policy



$$a = \pi_\theta(s)$$

e.g. outputs a steering angle directly

stochastic continuous policy



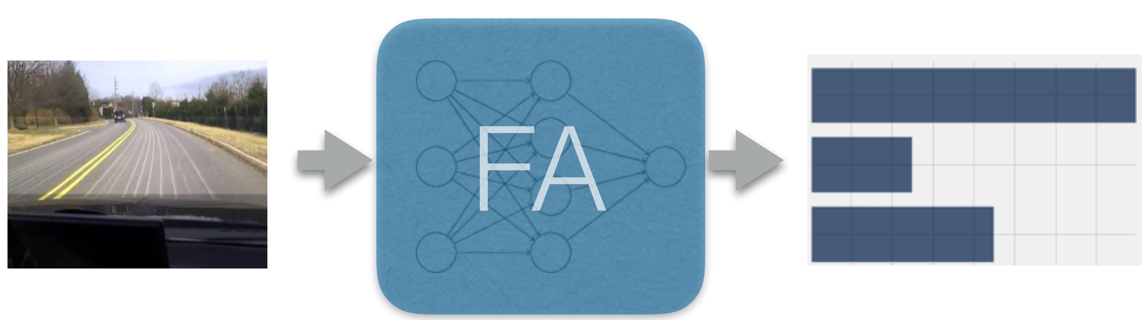
$\mu_\theta(s)$

$\sigma_\theta(s)$

$$a \sim \mathcal{N}(\mu_\theta(s), \sigma^2_\theta(s))$$

FA for stochastic multimodal continuous policies is an active area of research

(stochastic) policy over discrete actions



go left

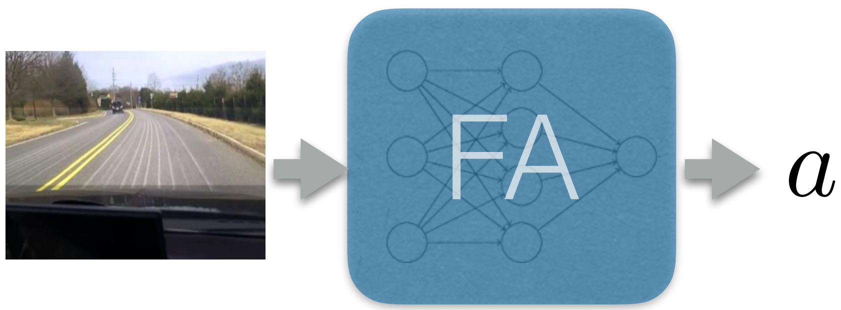
go right

press brake

Outputs a distribution over a discrete set of actions

Policy functions

deterministic continuous policy



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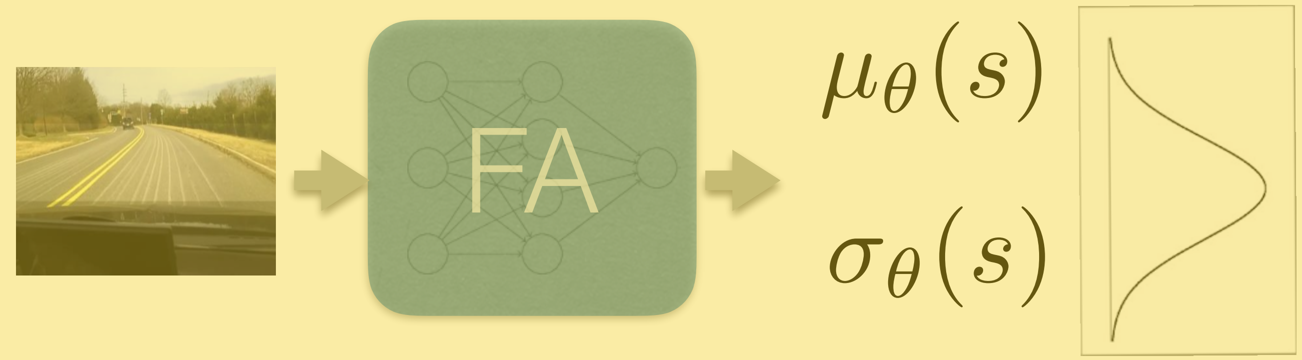

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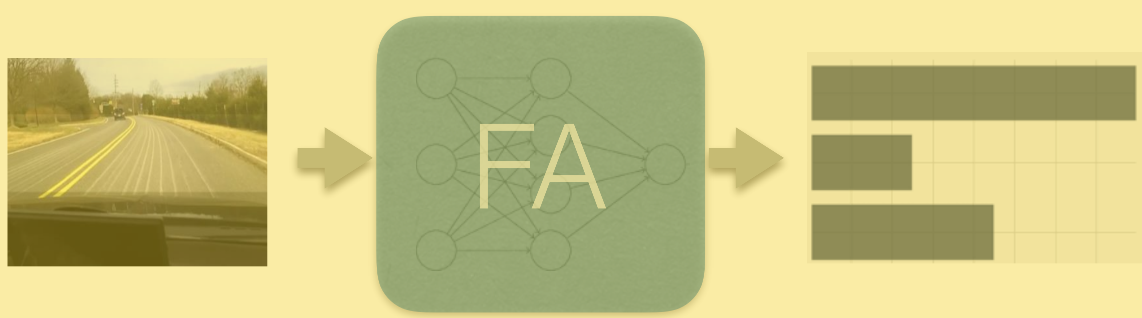

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Policy objective

Trajectory τ is a state action sequence $s_0, a_0, s_1, a_1, \ldots s_H, a_H$

Trajectory reward: $R(\tau) = \sum_{t=0}^{H} R(s_t, a_t)$

A reasonable policy objective then is:

$$\max_{\theta} . \quad U(\theta) = \mathbb{E}_{\tau \sim P(\tau;\theta)}[R(\tau)] = \sum_{\tau} P(\tau;\theta) R(\tau)$$

Probability of a trajectory: $P(\tau;\theta) = \prod_{t=0}^{H} \underbrace{P(s_{t+1} \mid s_t, a_t)}_{\text{dynamics}} \cdot \underbrace{\pi_\theta(a_t \mid s_t)}_{\text{policy}}$

Our problem is to compute $\nabla_\theta U(\theta) = \nabla_\theta \mathbb{E}_{\tau \sim P(\tau;\theta)}[R(\tau)]$

Policy objective

$$U(\theta) = \mathbb{E}_{\tau \sim P(\tau;\theta)}[R(\tau)]$$

Computing derivatives of expectations w.r.t. variables that parametrize the distribution, not the quantity inside the expectation

$$\max_{\theta} . \quad \mathbb{E}_{x \sim P(x;\theta)} f(x)$$

Assumptions:

- P is a probability density function that is continuous and differentiable
- P is easy to sample from

Derivatives of expectations

$$\nabla_\theta \mathbb{E}_x f(x) = \nabla_\theta \mathbb{E}_{x \sim P_\theta(x)} \left[f(x) \right]$$

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$$\begin{aligned}\nabla_\theta \mathbb{E}_x f(x) &= \nabla_\theta \mathbb{E}_{x\sim P_\theta(x)}\left[f(x)\right] \\ &= \nabla_\theta \sum_x P_\theta(x) f(x)\end{aligned}$$

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Why?

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What is the problem here?

Derivatives of expectations

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What have we achieved?

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I can obtain an unbiased estimator for the gradient $\nabla_\theta \mathbb{E}_x f(x)$ by sampling!

From the law of large numbers, it will converge to the right gradient with infinite number of samples

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Derivatives of the policy objective

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Approximate the gradient with empirical estimate from N sampled trajectories:

$$\nabla_\theta U(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \nabla_\theta \log P_\theta(\tau^{(i)}) R(\tau^{(i)})$$

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From trajectories to actions

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Let's call $\hat{g}$ the approximate gradient vector

$$\nabla_\theta U(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \nabla_\theta \log P_\theta(\tau^{(i)}) R(\tau^{(i)})$$

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$\nabla_\theta \log \pi_\theta(a)$ for Gaussian policy

Variance may be fixed, or can also be parameterized, for now let's fix it.

- For univariate Gaussian $\mathcal{N}(\mu, \sigma^2) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(a-\mu)^2}{\sigma^2}}$

$$\nabla_\theta \log \pi_\theta(a \mid s) = \text{const.} \cdot \frac{(a - \mu_\theta(s))}{\sigma^2} \nabla_\theta \mu_\theta(s)$$

- For multivariate Gaussian $\mathcal{N}(\mu, \Sigma) = \dfrac{1}{\sqrt{(2\pi)^k \det \Sigma}} e^{-\frac{1}{2}(\mathbf{a}-\mu)^\top \Sigma^{-1}(\mathbf{a}-\mu)}$

$$\nabla_\theta \log \pi_\theta(a \mid s) = \text{const.} \cdot \Sigma^{-1}(\mathbf{a} - \mu_\theta(s)) \frac{\partial(\mu_1 \ldots \mu_k)}{\partial(\theta_1 \ldots \theta_n)}$$

Two-dimensional Gaussian $P_\theta(x)$

$$
\begin{aligned}
\nabla_\theta \mathbb{E}_x f(x) &= \nabla_\theta \mathbb{E}_{x\sim P_\theta(x)}\left[f(x)\right] \\
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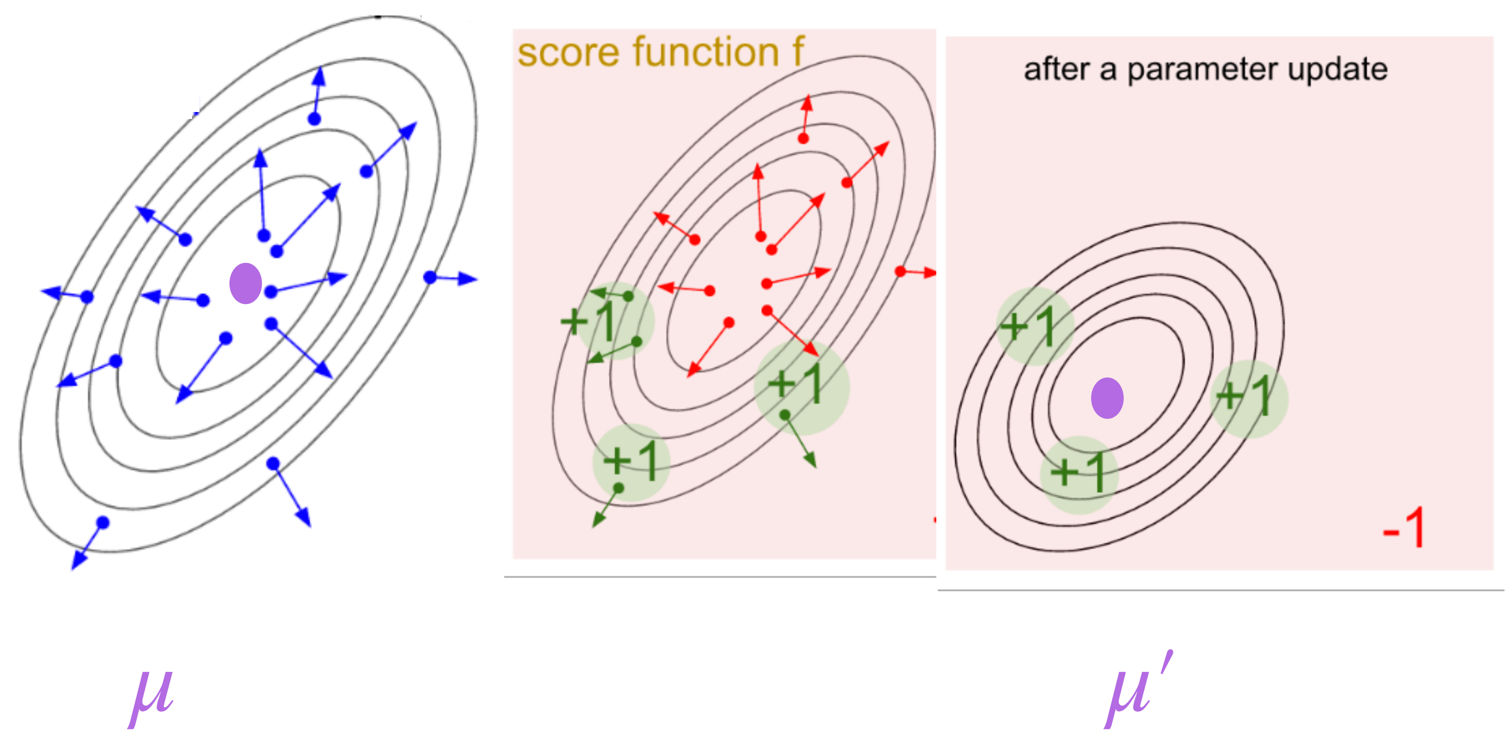
&= \mathbb{E}_{x\sim P_\theta(x)}\left[\nabla_\theta \log P_\theta(x) f(x)\right] \\
&\approx \frac{1}{N}\sum_{i=1}^{N} \nabla_\theta \log P_\theta(x^{(i)}) f(x^{(i)}) \\
&\approx \frac{1}{N}\sum_{i=1}^{N} \text{const.}\Sigma^{-1}\cdot(\mathbf{a}^{(i)}-\mu)\begin{bmatrix}10\\01\end{bmatrix} f(x^{(i)})
\end{aligned}
$$

Imagine I am optimizing over these two parameters:

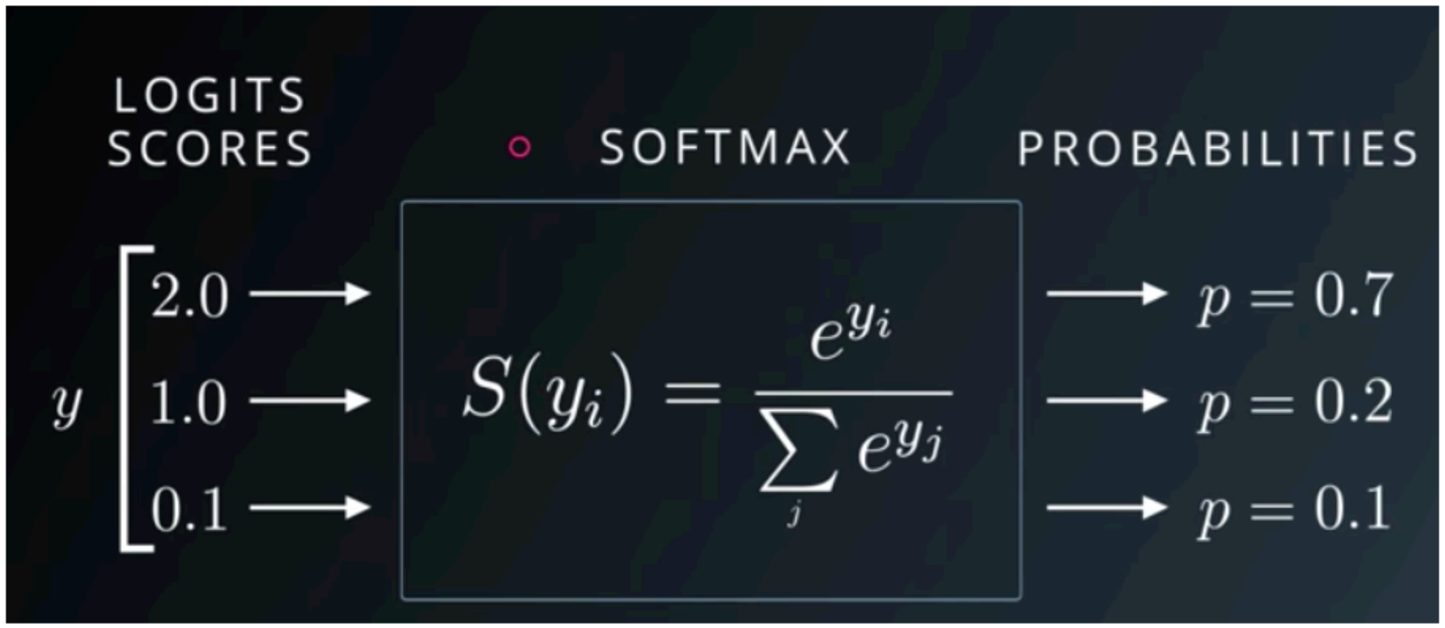
$$\mu = [\theta_1, \theta_2]^\top$$

- $\mathbf{a}^{(i)}$

→ $\Sigma^{-1}(\mathbf{a}^{(i)} - \mu) \propto \nabla_\theta \log P_\theta(\mathbf{a})$



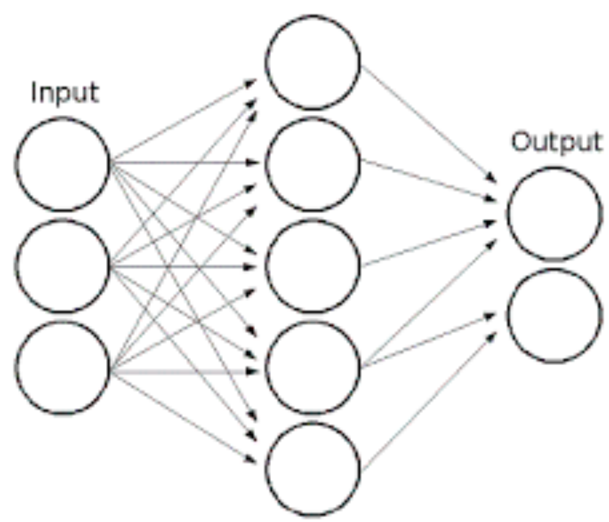
Softmax



$$\pi(a \mid s, \theta) \doteq \frac{e^{h(s,a,\theta)}}{\sum_b e^{h(s,b,\theta)}}$$

discrete actions

s



go left

go right

Output is a distribution over a discrete set of actions

$\nabla_\theta \log \pi_\theta(a)$ for softmax policy

$$\pi_\theta(a \mid s) = \frac{e^{h_\theta(s,a)}}{\sum_b e^{h_\theta(s,b)}}$$

$\nabla_\theta \log \pi_\theta(a) =$

$\nabla_\theta \log \pi_\theta(a)$ for softmax policy

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$$\nabla_\theta \log \pi_\theta(a) = \nabla_\theta h_\theta(s,a) - \nabla_\theta \log \sum_b e^{h_\theta(s,b)}$$

$\nabla_\theta \log \pi_\theta(a)$ for softmax policy

$$\pi_\theta(a \mid s) = \frac{e^{h_\theta(s,a)}}{\sum_b e^{h_\theta(s,b)}}$$

$$\begin{aligned}\nabla_\theta \log \pi_\theta(a) = \; & \nabla_\theta h_\theta(s,a) - \nabla_\theta \log \sum_b e^{h_\theta(s,b)} \\ & \nabla_\theta h_\theta(s,a) - \frac{1}{\sum_b e^{h_\theta(s,b)}} \nabla_\theta \sum_b e^{h_\theta(s,b)}\end{aligned}$$

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& \nabla_\theta h_\theta(s,a) - \frac{1}{\sum_b e^{h_\theta(s,b)}} \sum_b \nabla_\theta e^{h_\theta(s,b)}
\end{aligned}$$

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& \nabla_\theta h_\theta(s,a) - \frac{1}{\sum_b e^{h_\theta(s,b)}} \sum_b e^{h_\theta(s,b)} \nabla_\theta h_\theta(s,b)
\end{aligned}$$

$\nabla_\theta \log \pi_\theta(a)$ for softmax policy

$$\pi_\theta(a \mid s) = \frac{e^{h_\theta(s,a)}}{\sum_b e^{h_\theta(s,b)}}$$

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\end{aligned}$$

$\nabla_\theta \log \pi_\theta(a)$ for softmax policy

$$\pi_\theta(a \mid s) = \frac{e^{h_\theta(s,a)}}{\sum_b e^{h_\theta(s,b)}}$$

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& \nabla_\theta h_\theta(s, a) - \sum_b \frac{e^{h_\theta(s,b)}}{\sum_b e^{h_\theta(s,b)}} \nabla_\theta h_\theta(s, b) \\
& \nabla_\theta h_\theta(s, a) - \sum_b \pi_\theta(s, b) \nabla_\theta h_\theta(s, b)
\end{aligned}$$

Temporal structure

$$\hat{g} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t^{(i)} \mid s_t^{(i)}) R(\tau^{(i)})$$

Temporal structure

$$\hat{g} = \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\alpha_t^{(i)} \mid s_t^{(i)})R(\tau^{(i)})$$
$$= \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\alpha_t^{(i)} \mid s_t^{(i)})\left(\sum_{k=0}^{T}R(s_k^{(i)}, a_k^{(i)})\right)$$

Each action takes the blame for the full trajectory!

Temporal structure

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$$= \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(a_t^{(i)} \mid s_t^{(i)})\left(\sum_{k=0}^{t-1} R(s_k^{(i)}, a_k^{(i)}) + \sum_{k=t}^{T} R(s_k^{(i)}, a_k^{(i)})\right)$$

Each action takes the blame for the full trajectory!

These rewards are not caused by actions that come after t

Can we do better than assigning the cumulative trajectory reward to every action in the trajectory?

Temporal structure

$$\hat{g} = \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\alpha_t^{(i)} \mid s_t^{(i)})R(\tau^{(i)})$$

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Each action takes the blame for the full trajectory!

Consider instead:

$$\hat{g} = \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\alpha_t^{(i)} \mid s_t^{(i)})\left(\sum_{k=t}^{T} R(s_k^{(i)}, a_k^{(i)})\right)$$

Each action takes the blame for the trajectory that comes after it

We can call this the return from t onwards G_t

Likelihood ratio gradient estimator

- Let's analyze the update:

$$\Delta\theta_t = \alpha G_t \nabla_\theta \log \pi_\theta \left(s_t, a_t\right)$$

- Let's us rewrite is as follows:

move most in the directions that favor actions that yield the highest return

$$\boldsymbol{\theta}_{t+1} \doteq \boldsymbol{\theta}_t + \alpha\gamma^t G_t \frac{\nabla_{\boldsymbol{\theta}}\pi\left(A_t | S_t, \boldsymbol{\theta}\right)}{\pi\left(A_t | S_t, \boldsymbol{\theta}\right)}$$

Update is inversely proportional to the action probability to fight the fact that actions that are selected frequently are at an advantage (the updates will be more often in their direction)

REINFORCE (or Monte Carlo Policy Gradient)

0. Initialize policy parameters θ

1. Sample trajectories $\{\tau_i = \{s_t^i, a_t^i\}_{t=0}^T\}$ by deploying the current policy $\pi_\theta(a_t \mid s_t)$.

2. Compute gradient vector $\nabla_\theta U(\theta) \approx \hat{g} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(a_t^{(i)} \mid s_t^{(i)}) G_t^{(i)}$

3. $\theta \leftarrow \theta + \alpha \nabla_\theta U(\theta)$

Policy Gradients

$$\max_{\theta}. \; U(\theta) = \mathbb{E}_{\tau \sim P_\theta(\tau)} \left[R(\tau) \right]$$

$$\begin{aligned}
\nabla_\theta U(\theta) &= \nabla_\theta \mathbb{E}_{\tau \sim P_\theta(\tau)} \left[R(\tau) \right] \\
&= \nabla_\theta \sum_\tau P_\theta(\tau) R(\tau) \\
&= \sum_\tau \nabla_\theta P_\theta(\tau) R(\tau) \\
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&= \sum_\tau P_\theta(\tau) \nabla_\theta \log P_\theta(\tau) R(\tau) \\
&= \mathbb{E}_{\tau \sim P_\theta(\tau)} \left[\nabla_\theta \log P_\theta(\tau) R(\tau) \right]
\end{aligned}$$

Sample estimate:

$$\nabla_\theta U(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \nabla_\theta \log P_\theta(\tau^{(i)}) R(\tau^{(i)})$$

Evolutionary Methods

$$\max_{\mu}.\ U(\mu) = \mathbb{E}_{\theta \sim P_\mu(\theta)}\left[F(\theta)\right]$$

$$\begin{aligned}
\nabla_\mu U(\mu) &= \nabla_\mu \mathbb{E}_{\theta \sim P_\mu(\theta)}\left[F(\theta)\right] \\
&= \nabla_\mu \int P_\mu(\theta) F(\theta) d\theta \\
&= \int \nabla_\mu P_\mu(\theta) F(\theta) d\theta \\
&= \int P_\mu(\theta) \frac{\nabla_\mu P_\mu(\theta)}{P_\mu(\theta)} F(\theta) d\theta \\
&= \int P_\mu(\theta) \nabla_\mu \log P_\mu(\theta) F(\theta) d\theta \\
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\end{aligned}$$

Sample estimate:

$$\nabla_\mu U(\mu) \approx \frac{1}{N} \sum_{i=1}^{N} \nabla_\mu \log P_\mu(\theta^{(i)}) F(\theta^{(i)})$$

Policy gradients VS Evolutionary methods

Considers distribution over actions

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&= \mathbb{E}_{\theta \sim P_\mu(\theta)} \left[\nabla_\mu \log P_\mu(\theta) F(\theta) \right]
\end{aligned}$$

Sample estimate:

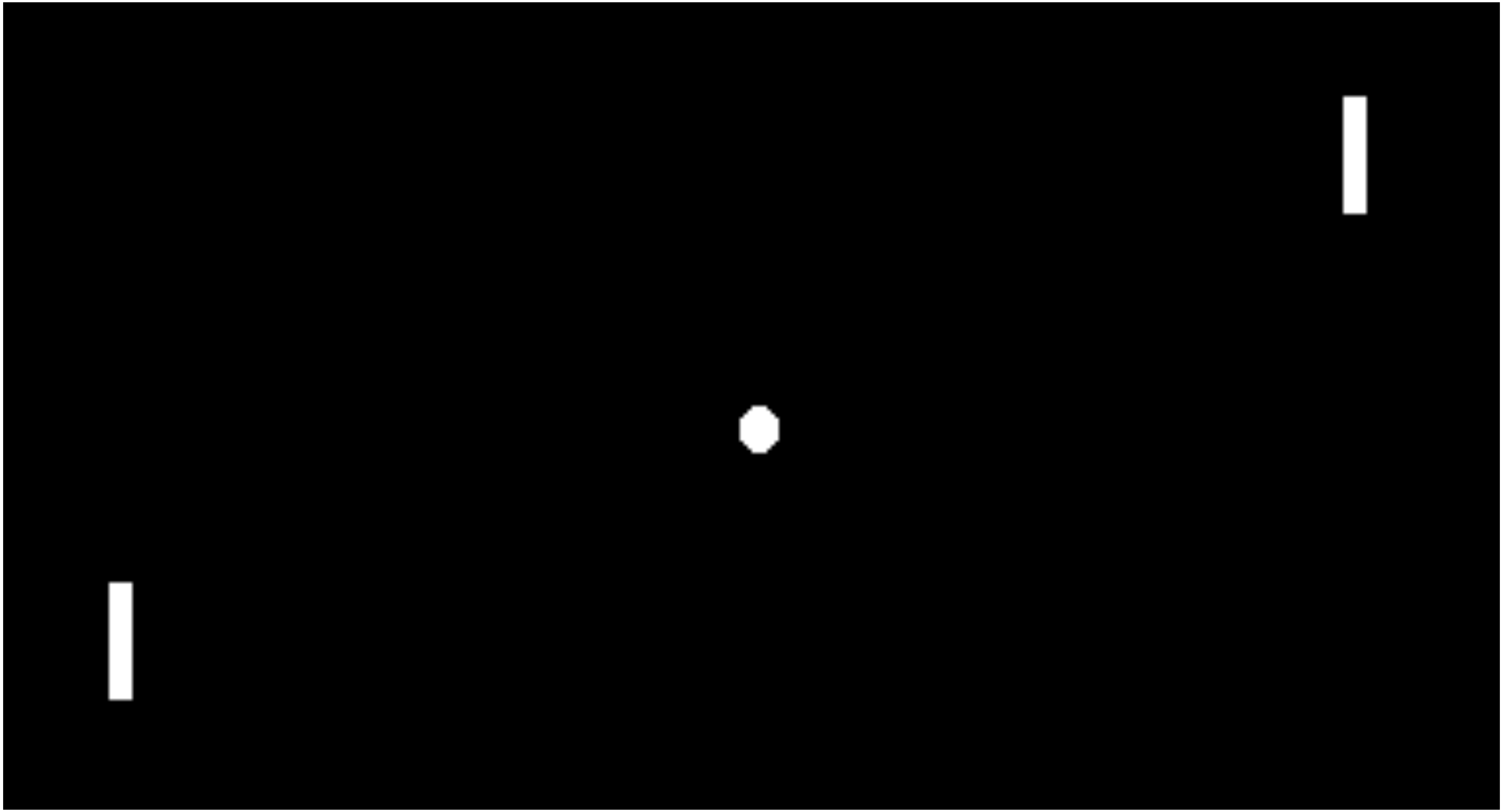
$$\nabla_\mu U(\mu) \approx \frac{1}{N} \sum_{i=1}^{N} \nabla_\mu \log P_\mu(\theta^{(i)}) F(\theta^{(i)})$$

Policy gradients VS Evolutionary methods

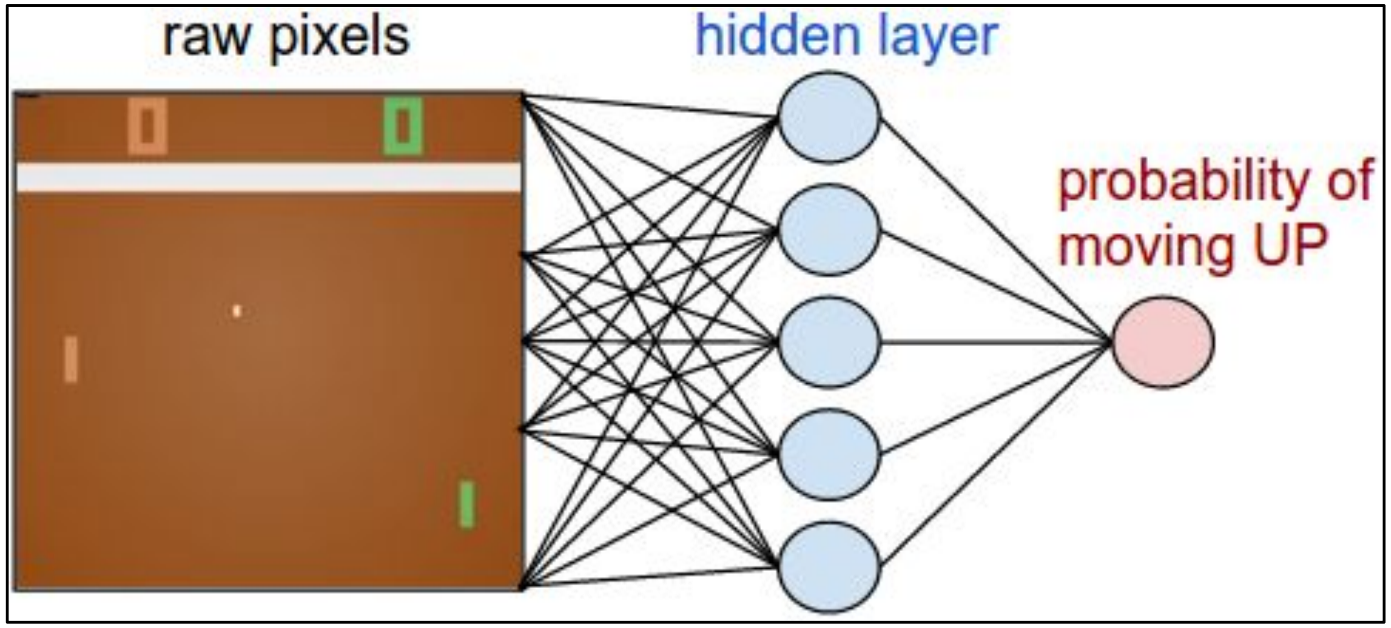
- We are sampling in both cases…
  - PG: sampling in action space
  - ES: sampling in parameter space

Pong from Pixels

Slides from Andrei Karpathy



Policy network

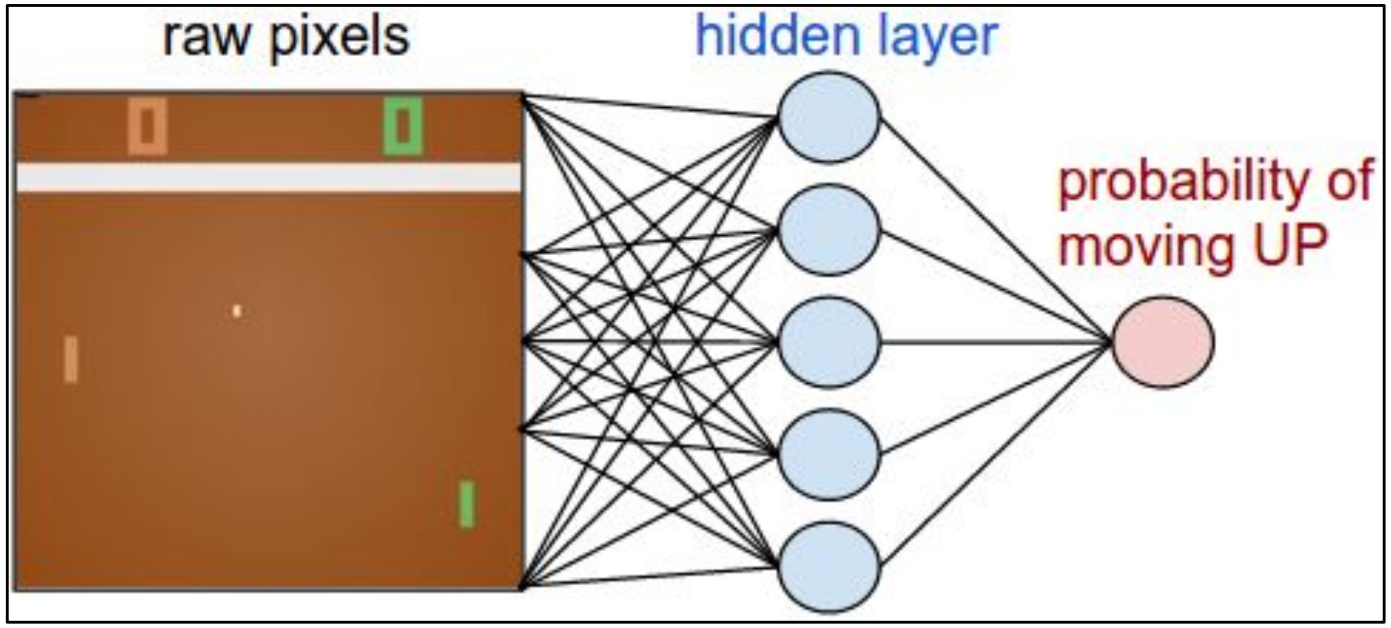


Policy network

e.g.,

height width

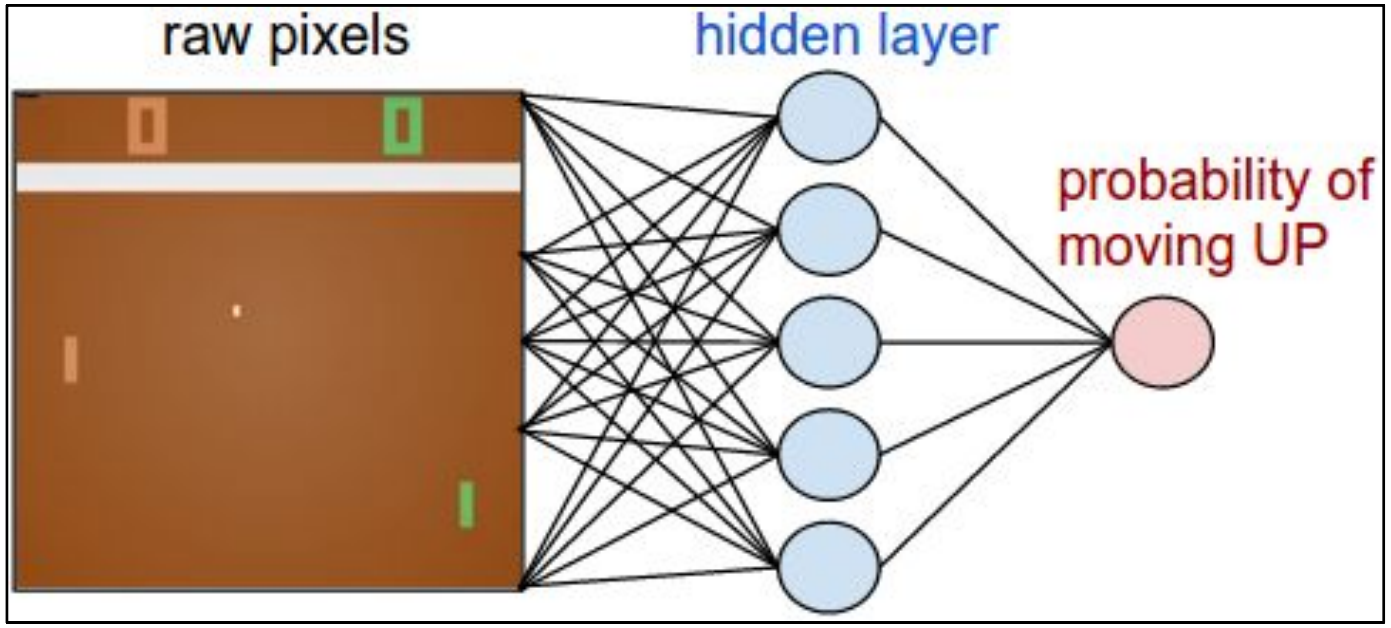
[80 x 80]
array



Policy network

height width

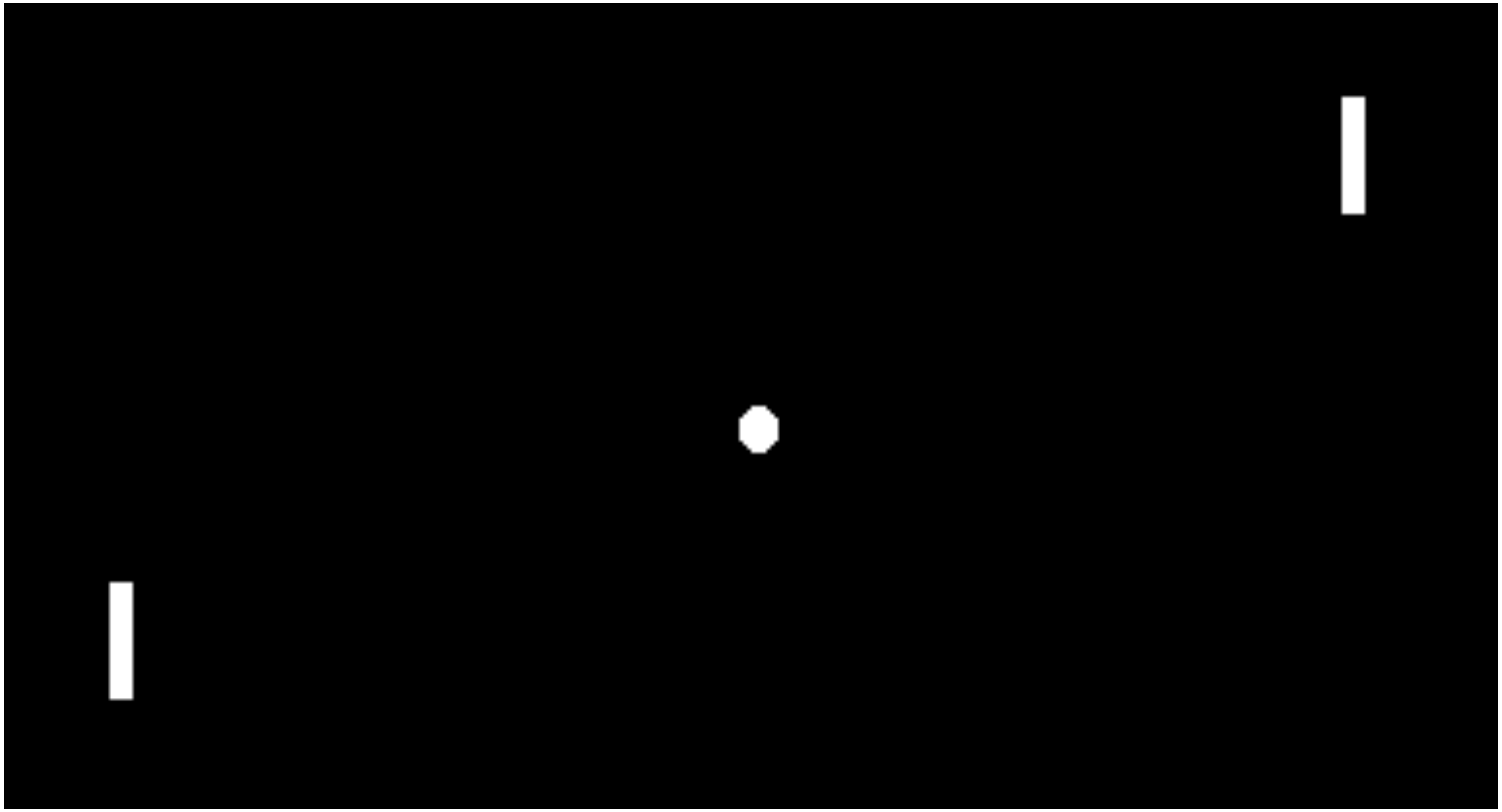
[**80** x **80**]
array



E.g. 200 nodes in the hidden network, so:

[(80*80)*200 + 200] + [200*1 + 1] = **~1.3M parameters**

Layer 1 Layer 2



Network does not see this. Network sees 80*80 = 6,400 numbers.
It gets a reward of +1 or -1, some of the time.
Q: How do we efficiently find a good setting of the 1.3M parameters?

Suppose we had the training labels…
(we know what to do in any state)

(x1,UP)
(x2,DOWN)
(x3,UP)
...

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(we know what to do in any state)

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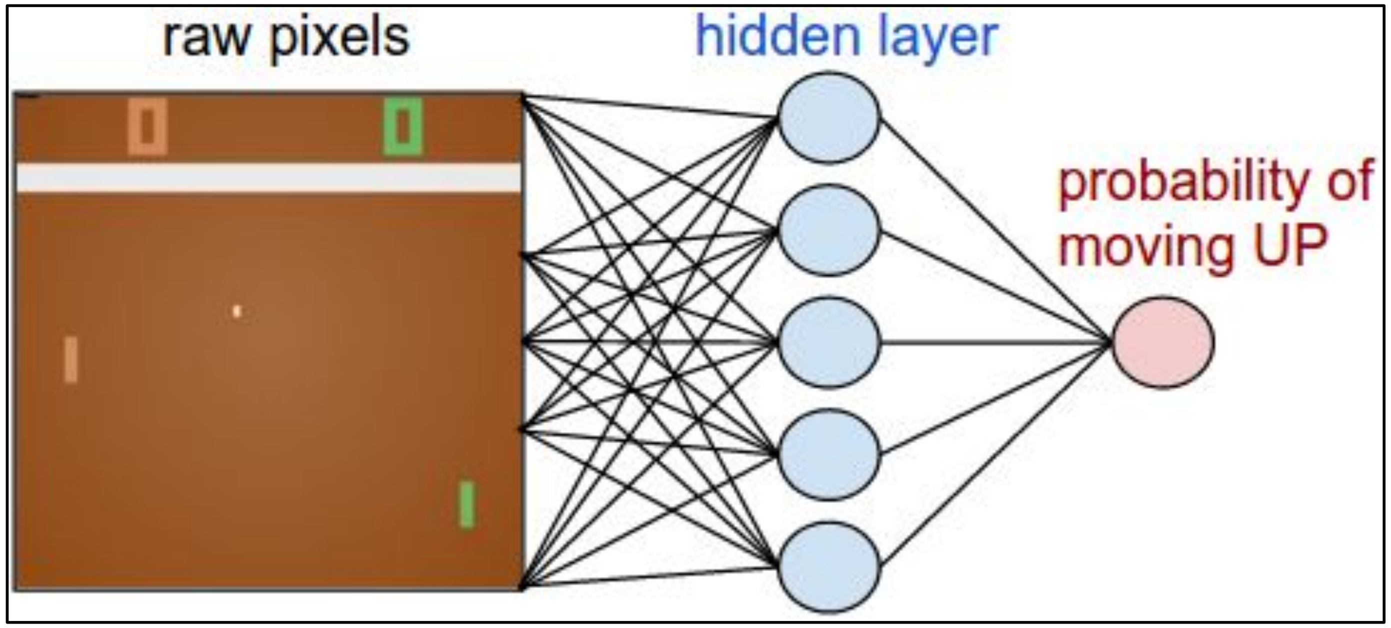
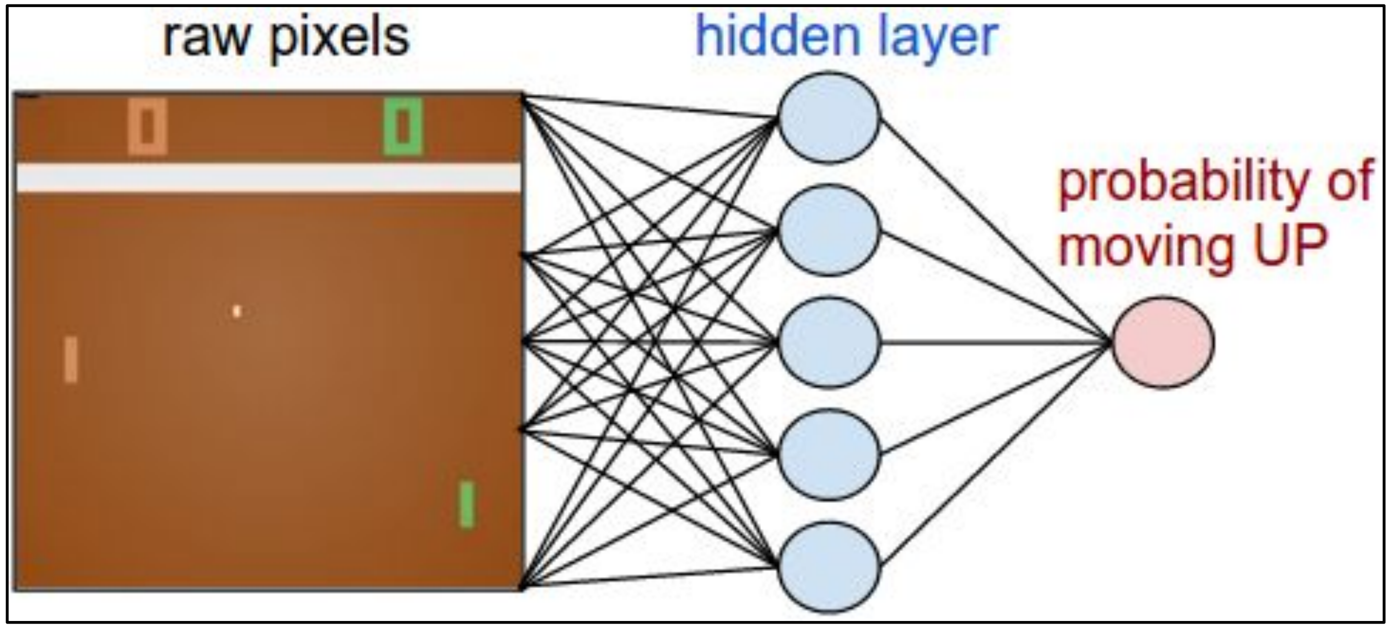


Suppose we had the training labels…
(we know what to do in any state)

(x1,UP)
(x2,DOWN)
(x3,UP)
...

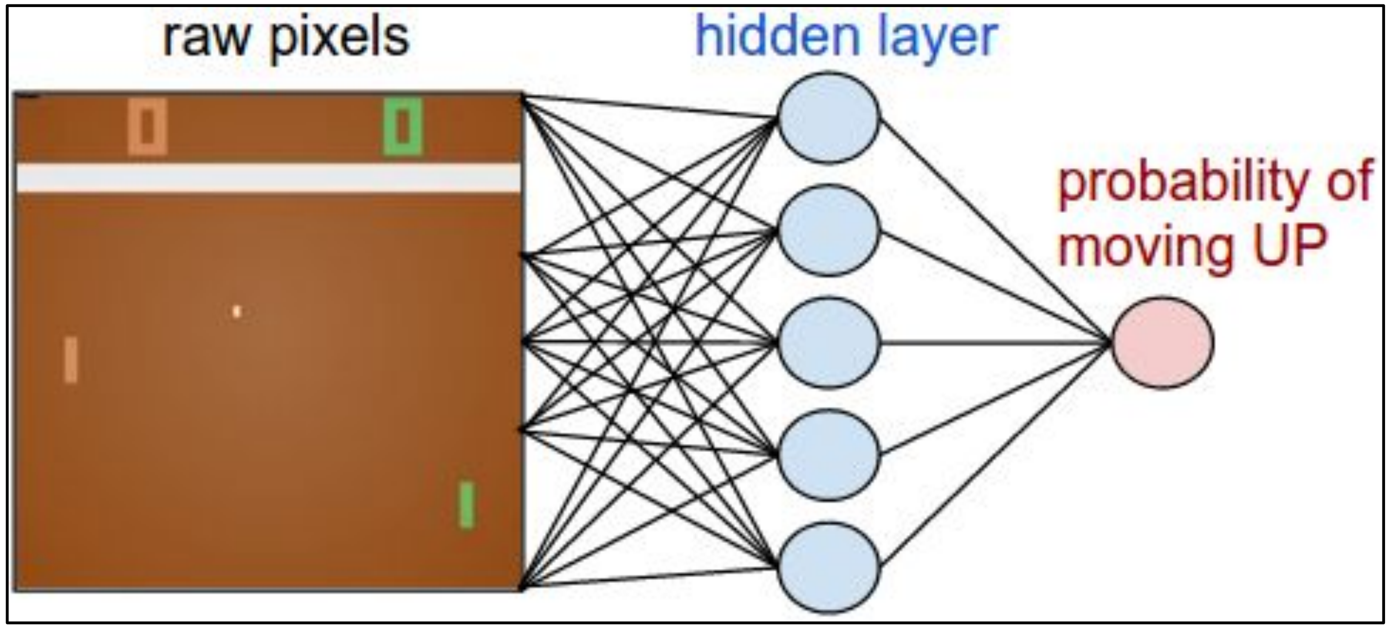
maximize:

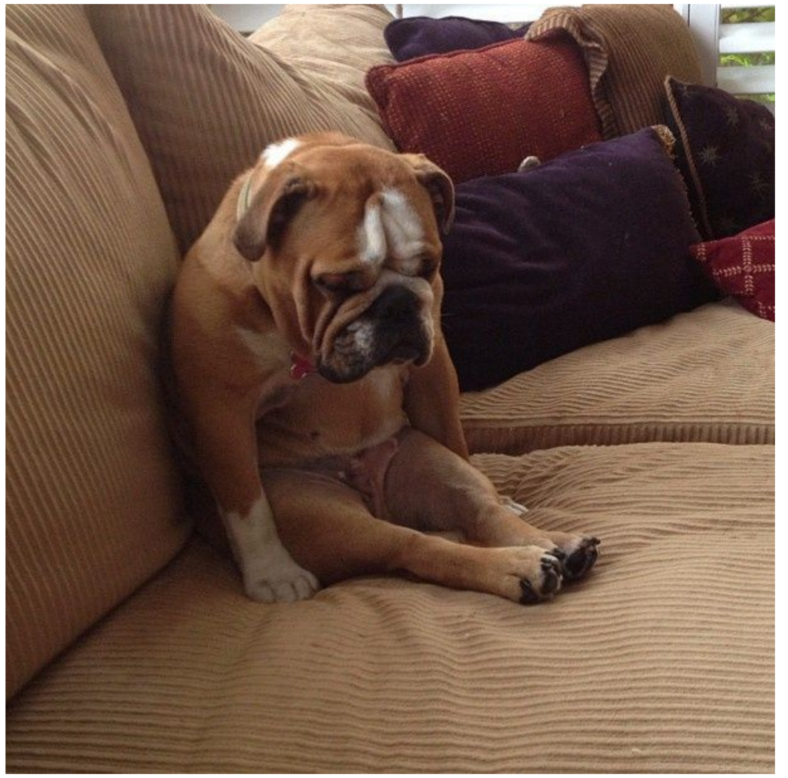
$$\sum_i \log p(y_i \,|\, x_i)$$



supervised learning

Except, we don't have labels...





Should we go UP or DOWN?

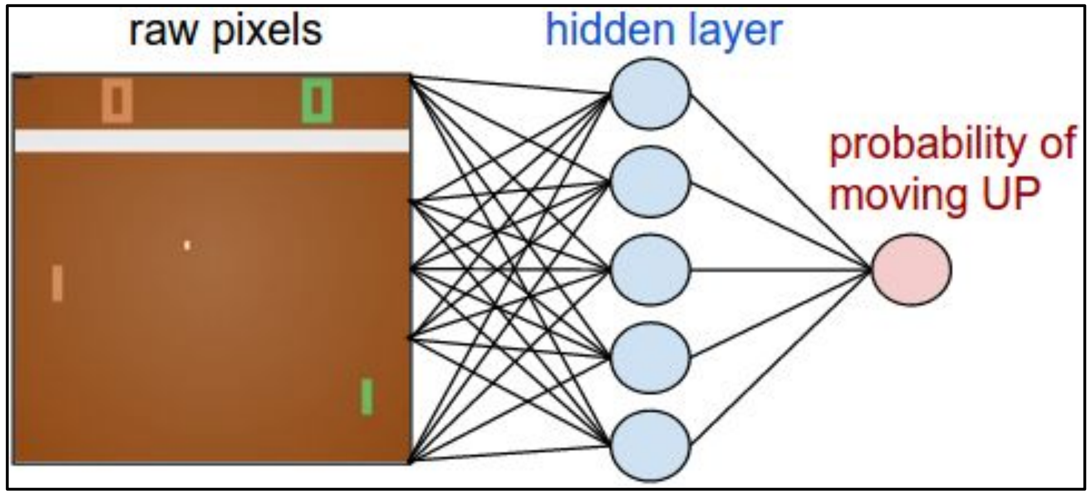
Except, we don’t have labels...

“Try a bunch of stuff and see what happens. Do more of the stuff that worked in the future.”

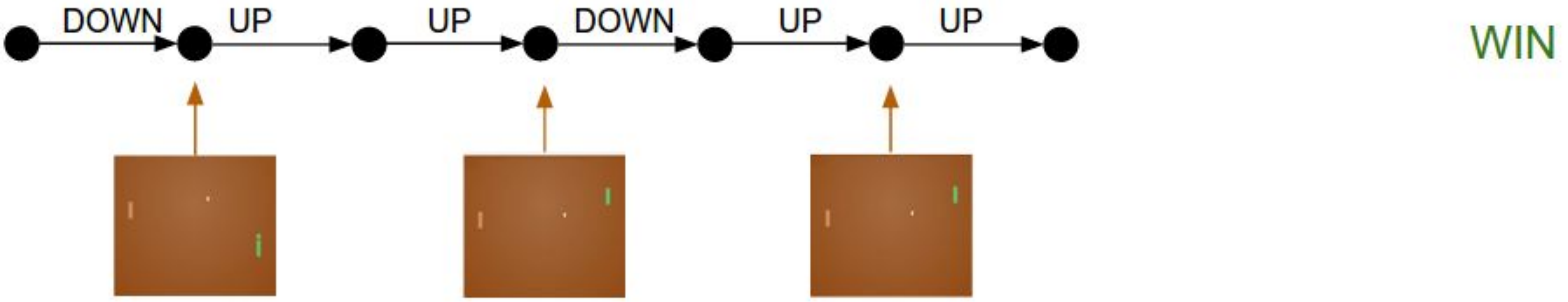
-RL

trial-and-error learning

Let's just act according to our current policy...

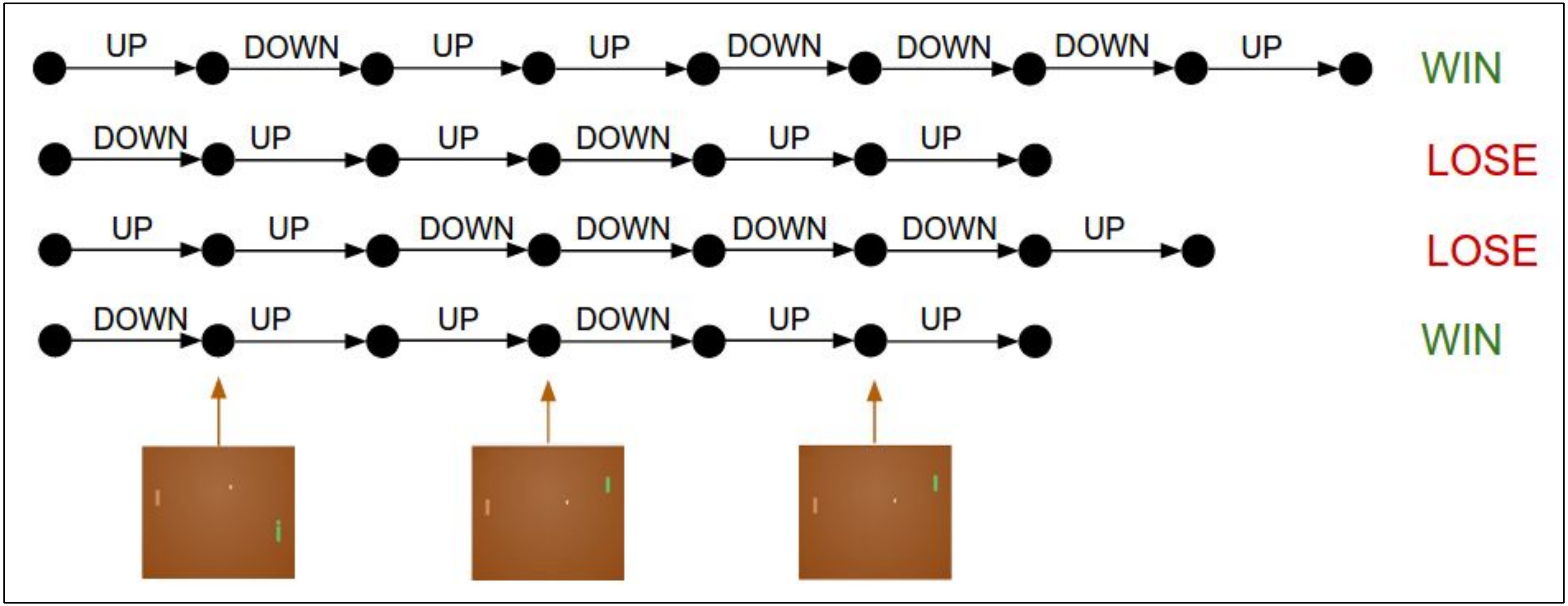


Rollout the policy and collect an episode

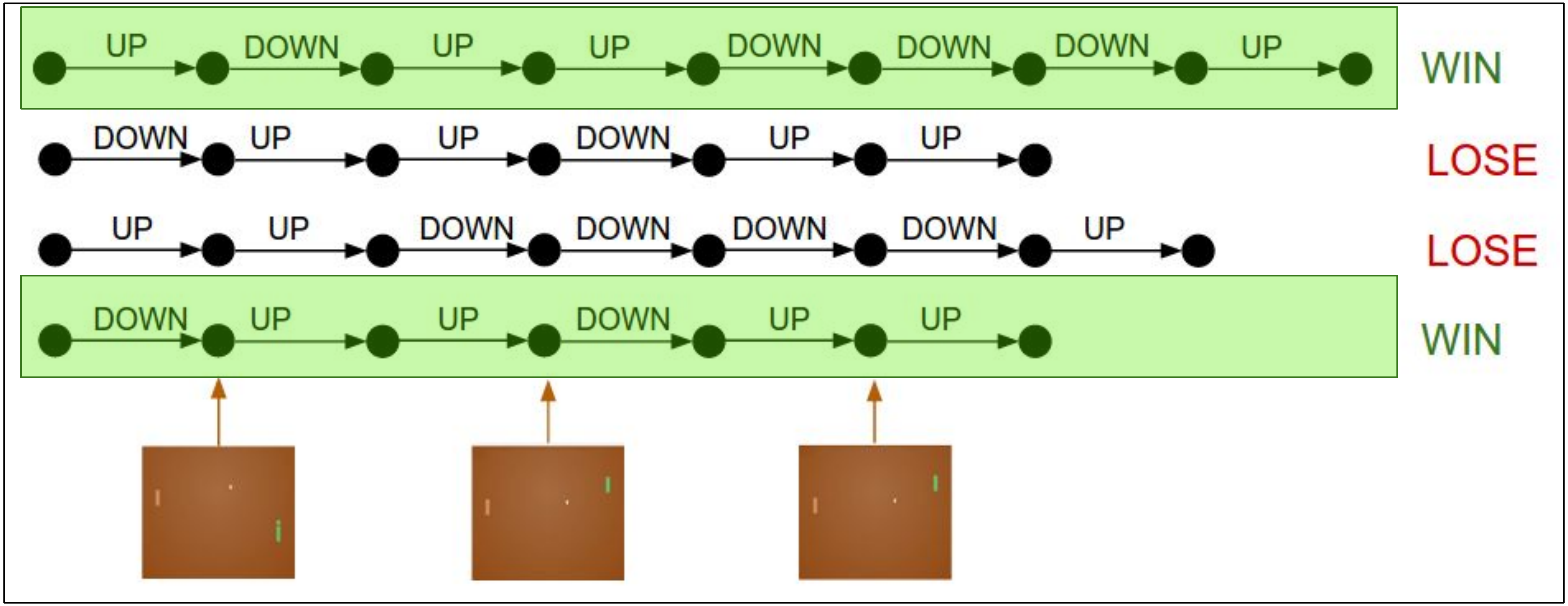


Collect many rollouts...

4 rollouts:



Not sure whatever we did here, but apparently it was good.



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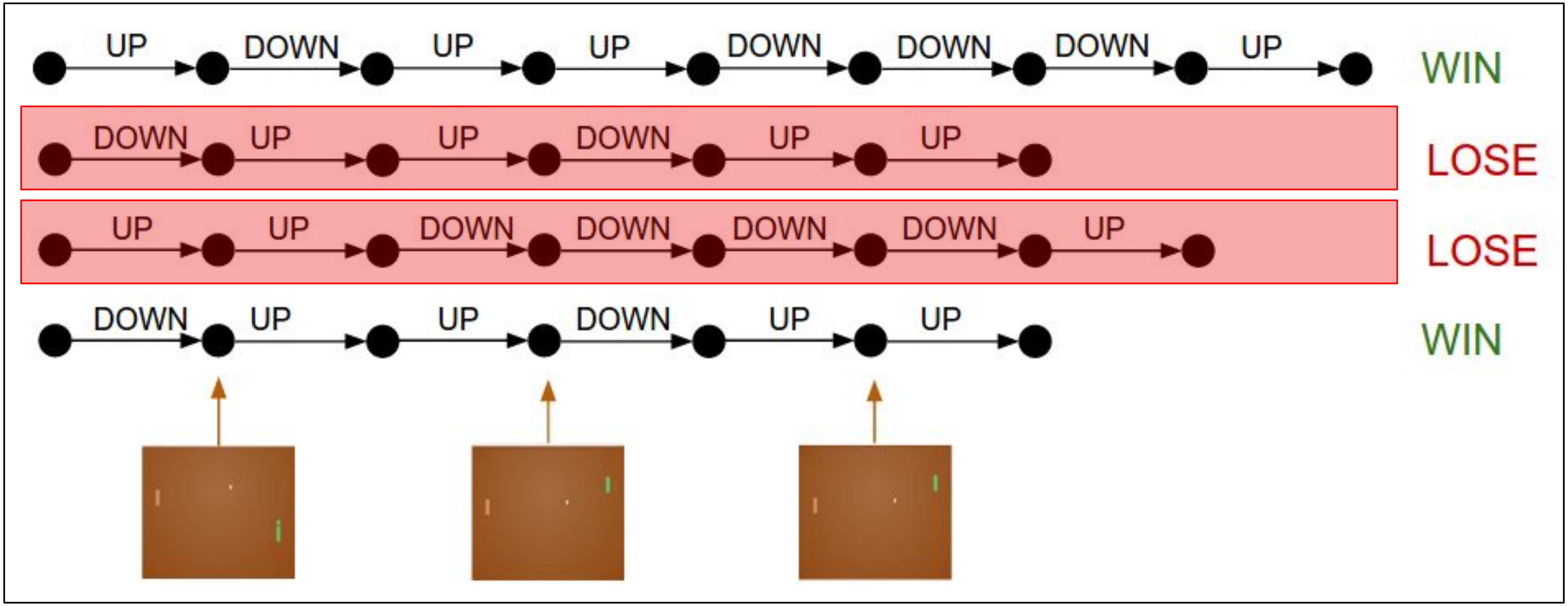
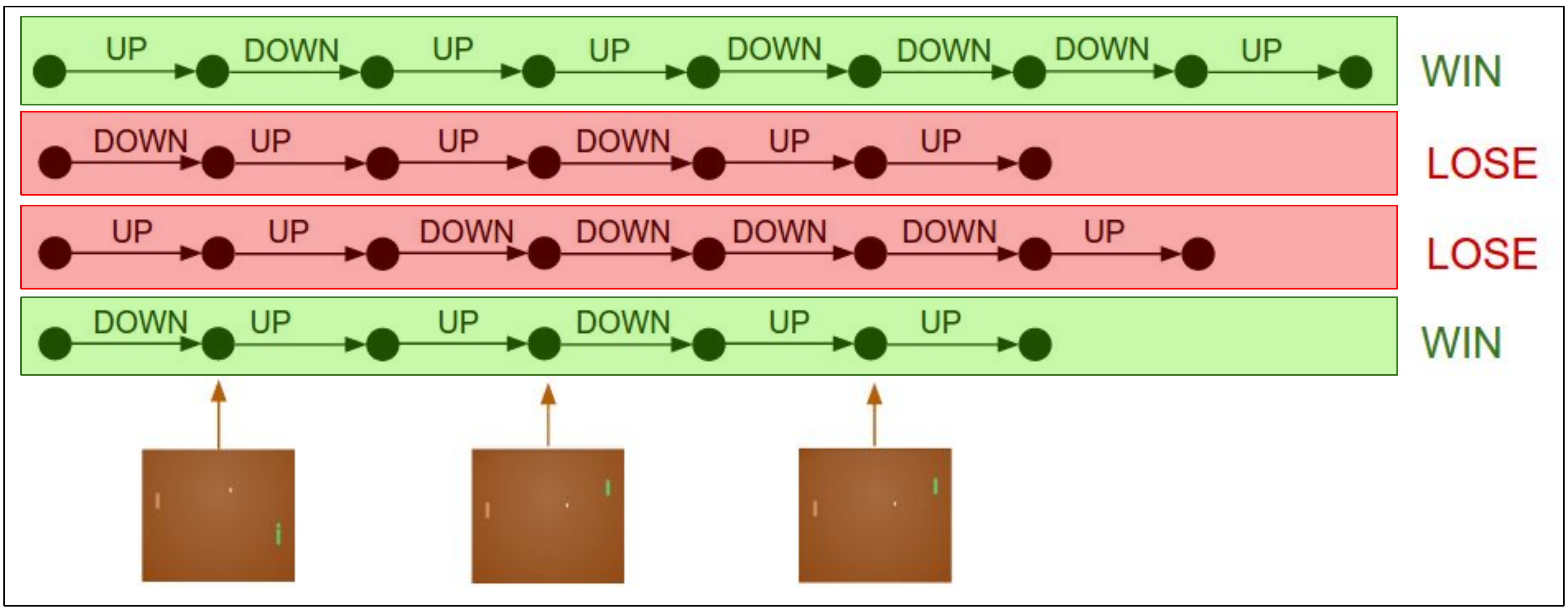


Pretend every action we took here was the correct label.

maximize: $\log p(y_i \mid x_i)$

Pretend every action we took here was the wrong label.

maximize: $(-1) * \log p(y_i \mid x_i)$



$$\nabla_\theta U(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t^{(i)} \mid s_t^{(i)}) R(\tau^{(i)})$$

Supervised Learning

maximize:

$$\sum_i \log p(y_i | x_i)$$

For images x_i and their labels y_i.

Supervised Learning

maximize:

$$\sum_i \log p(y_i | x_i)$$

For images x_i and their labels y_i.

Reinforcement Learning

Supervised Learning

maximize:

$$\sum_i \log p(y_i | x_i)$$

For images x_i and their labels y_i.

Reinforcement Learning

1) we have no labels so we sample:

$$y_i \sim p(\cdot | x_i)$$

Supervised Learning

maximize:

$$\sum_i \log p(y_i | x_i)$$

For images x_i and their labels y_i.

Reinforcement Learning

1) we have no labels so we sample:

$$y_i \sim p(\cdot | x_i)$$

2) once we collect a batch of rollouts:

maximize:

$$\sum_i A_i * \log p(y_i | x_i)$$

Supervised Learning

maximize:

$$\sum_i \log p(y_i | x_i)$$

For images x_i and their labels y_i.

Reinforcement Learning

1) we have no labels so we sample:

$$y_i \sim p(\cdot | x_i)$$

2) once we collect a batch of rollouts:

maximize:

$$\sum_i A_i * \log p(y_i | x_i)$$

We call this the **advantage**, it's a number, like +1.0 or -1.0 based on how this action eventually turned out.

Advantage is the same for all actions taken during a trajectory, and depends on the trajectory return (episode return)

Supervised Learning

maximize:

$$\sum_i \log p(y_i | x_i)$$

For images x_i and their labels y_i.

Reinforcement Learning

1) we have no labels so we sample:

$$y_i \sim p(\cdot | x_i)$$

2) once we collect a batch of rollouts:

maximize:

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+ve advantage will make that action more likely in the future, for that state.

-ve advantage will make that action less likely in the future, for that state.

$$\nabla_\theta U(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t^{(i)} | s_t^{(i)}) R(\tau^{(i)})$$

Variance

Here is our gradient estimator:

$$\hat{g} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t^{(i)} \,|\, s_t^{(i)}) G_t^{(i)}$$

It is unbiased, i.e., with large N it will accurately approximate the true gradient.

Problem: Unfortunately, usually a very large N is required.

Can we improve the variance of our estimator?

Variance

Here is our gradient estimator:

$$\hat{g} = \frac{1}{N}\sum_{i=1}^{N}\sum_{t=0}^{T}\nabla_\theta \log \pi_\theta(\alpha_t^{(i)} \mid s_t^{(i)})G_t^{(i)}$$

Variance is the trace of the covariance matrix:

$$\mathrm{Var}(\hat{g}) = \mathrm{tr}\left(\mathbb{E}\left[(\hat{g} - \mathbb{E}[\hat{g}])(\hat{g} - \mathbb{E}[\hat{g}])^T\right]\right) = \sum_{k=1}^{n}\mathbb{E}\left[\left(\hat{g}_k - \mathbb{E}[\hat{g}_k]\right)^2\right]$$

Our goal is to minimize the variance of our estimator.

Reducing variance by subtracting a baseline

What if we subtract a constant b from the rewards:

$$\hat{g} = \frac{1}{N}\sum_{i=1}^{N} \nabla_\theta \log P_\theta(\tau^{(i)})(R(\tau^{(i)}) - b)$$

$$\hat{g} = \frac{1}{N}\sum_{i=1}^{N} \nabla_\theta \log P_\theta(\tau^{(i)})R(\tau^{(i)}) - \frac{1}{N}\sum_{i=1}^{N} \nabla_\theta \log P_\theta(\tau^{(i)})b$$

Reducing variance by subtracting a baseline

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original formulation

new term

Reducing variance by subtracting a baseline

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original formulation

new term

$$\begin{aligned}\sum_\tau P(\tau;\theta)\nabla_\theta \log P_\theta(\tau)b &= \sum_\tau P(\tau;\theta)\frac{\nabla_\theta P_\theta(\tau)}{P(\tau;\theta)}b \\ &= \sum_\tau \nabla_\theta P_\theta(\tau)b \\ &= b\left(\sum_\tau \nabla_\theta P_\theta(\tau)\right) \\ &= b\left(\nabla_\theta \sum_\tau P_\theta(\tau)\right) \\ &= 0\end{aligned}$$

Reducing variance by subtracting a baseline

What if we subtract a constant b from the rewards:

$$\hat{g} = \frac{1}{N}\sum_{i=1}^{N} \nabla_\theta \log P_\theta(\tau^{(i)})(R(\tau^{(i)}) - b)$$

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new term

$$\begin{aligned}
\sum_\tau P(\tau;\theta)\nabla_\theta \log P_\theta(\tau)b &= \sum_\tau P(\tau;\theta)\frac{\nabla_\theta P_\theta(\tau)}{P(\tau;\theta)}b \\
&= \sum_\tau \nabla_\theta P_\theta(\tau)b \\
&= b\left(\sum_\tau \nabla_\theta P_\theta(\tau)\right) \\
&= b\left(\nabla_\theta \sum_\tau P_\theta(\tau)\right) \\
&= 0
\end{aligned}$$

- new term equals to 0 in expectation
- We are still unbiased

Baseline choices

Constant Baseline (single scalar): $b = \mathbb{E}[R(\tau)]$

$$\hat{g} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t^{(i)} | s_t^{(i)}) \left(G_t^{(i)} - b\right)$$

Time-dependent Baseline (a vector length T): $b_t = \sum_{i=1}^{N} G_t^{(i)}$

$$\hat{g} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t^{(i)} | s_t^{(i)}) \left(G_t^{(i)} - b_t\right)$$

State-dependent Baseline (a function):

$$b(s) = \mathbb{E}\left[r_t + r_{t+1} + r_{t+2} + \ldots + r_{T-1} \,\middle|\, s_t = s\right] = V_\pi(s)$$

$$\hat{g} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^{(i)} | s_t^{(i)})(G_t^{(i)} - b(s_t^{(i)}))$$

Variance

Subtracting a state-dependent Baseline:

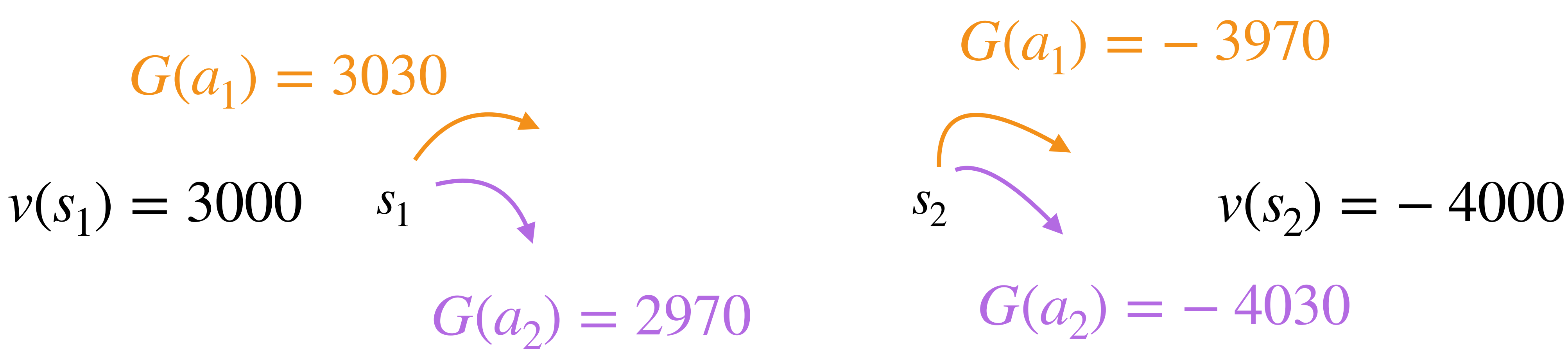
$$\hat{g} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^{(i)} \mid s_t^{(i)}) G_t^{(i)}$$

$$\text{Var}(\hat{g}) = \text{tr}\left(\mathbb{E}\left[(\hat{g} - \mathbb{E}[\hat{g}])(\hat{g} - \mathbb{E}[\hat{g}])^T\right]\right) = \sum_{k=1}^{n} \mathbb{E}\left[\left(\hat{g}_k - \mathbb{E}[\hat{g}_k]\right)^2\right]$$

- Imagine in some state S_1 the rewards of all actions are ~3000 and in some state S_2 the rewards of all actions are~ -4000.

Action advantages Versus Action returns

$$\hat{g} = \frac{1}{N}\sum_{i=1}^{N}\sum_{t=0}^{T}\nabla_\theta \log \pi_\theta(a_t^{(i)} \mid s_t^{(i)})\underbrace{(G_t^{(i)} - b(s_t^{(i)}))}_{\text{advantage}}$$



- Imagine in some state s_1 the rewards of all actions are ~3000 and in some state s_2 the rewards of all actions are~ -4000. Bad actions in s_1 will much influence the gradient way more than good actions in s_2. But our goal is to select the right actions across both bad and good states!

- Now imagine you have $b(s_1) = 3000$ and $b(s_2) = -4000$ and you subtract those values from the sample returns.

Action advantages Versus Action returns

$$\hat{g} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^{(i)} \mid s_t^{(i)}) \underbrace{(G_t^{(i)} - b(s_t^{(i)}))}_{\text{advantage}}$$

$G(a_1) = -3970$

$v(s_1) = 3000$ s_1 s_2 $v(s_2) = -4000$

$G(a_2) = 2970$

- Imagine in some state s_1 the rewards of all actions are ~3000 and in some state s_2 the rewards of all actions are~ -4000. Bad actions in s_1 will much influence the gradient way more than good actions in s_2. But our goal is to select the right actions across both bad and good states!
- Now imagine you have $b(s_1) = 3000$ and $b(s_2) = -4000$ and you subtract those values from the sample returns.
- We want to encourage an action not when it has high return, but when it has higher return than the other actions from that state, i.e., when it has an advantage over other actions. It may well be that a state is bad and all actions have low returns in that state, we care to find the actions that have higher returns that the rest, and we need to calibrate for the goodness of state using state dependent baselines.

Estimate state dependent baseline $V^{\pi}_{\phi}(s_t)$

$$\hat{g} = \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_{\theta}\log\pi_{\theta}(\alpha_t^{(i)}\mid s_t^{(i)})\left(G_t^{(i)} - V^{\pi}_{\phi}(s_t^{(i)})\right)$$

- Monte Carlo estimation

1. Initialize ϕ
2. Collect trajectories: $\tau_1, \ldots, \tau_N$
3. Regress against empirical return, e.g., using gradient descent:

$$\phi \leftarrow \arg\min_{\phi}\frac{1}{N}\sum_{i=1}^{N}\sum_{t=0}^{T-1}\left(V^{\pi}_{\phi}\left(s_t^{(i)}\right) - \left(\sum_{k=t}^{T-1} R\left(s_k^{(i)}, a_k^{(i)}\right)\right)\right)^2$$

Estimate state dependent baseline $V^{\pi}_{\phi}(s_t)$

$$\hat{g} = \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_{\theta}\log \pi_{\theta}(\alpha_t^{(i)} \mid s_t^{(i)})\left(G_t^{(i)} - V^{\pi}_{\phi}(s_t^{(i)})\right)$$

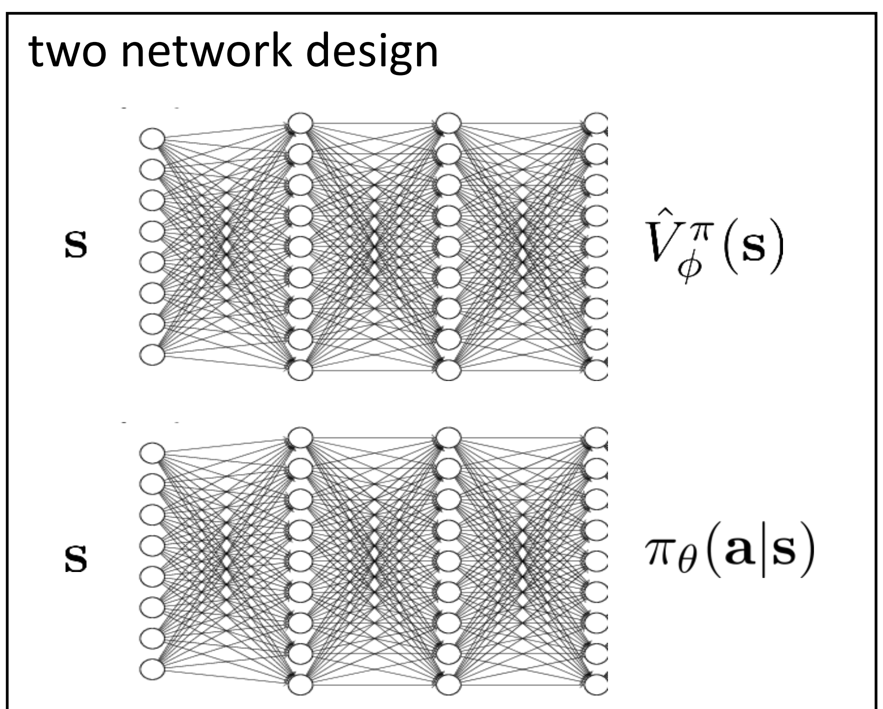
- TD estimation

1. Initialize ϕ
2. Collect data for: s, a, s', r
3. Fitted V iteration

$$\phi_{\ell+1} \leftarrow \arg\min_{\phi} \sum_{(s,a,s',r)} \left\| (r + V^{\pi}_{\phi_\ell}(s')) - V_{\phi}(s) \right\|_2^2$$

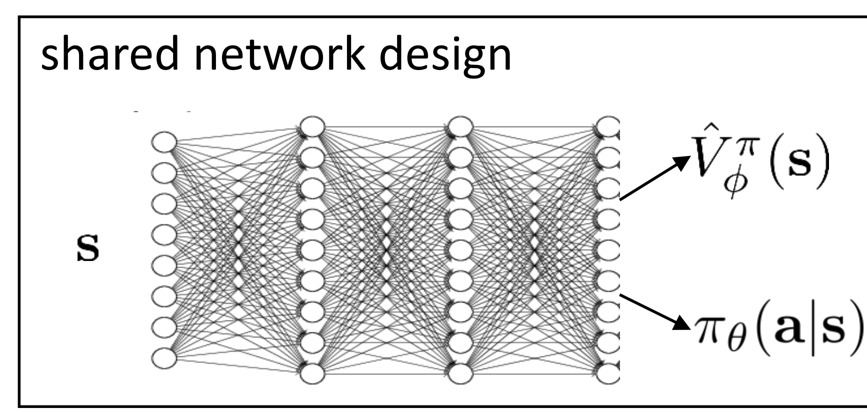
Actor-Critic

- Actor-critic algorithms maintain two sets of parameters
  - Critic Updates action-value function parameters w
  - Actor Updates policy parameters θ, in direction suggested by critic



+ simple & stable

- no shared features between actor & critic



Actor-Critic

0. Initialize policy parameters θ and critic parameters ϕ .

1. Sample trajectories $\{\tau_i = \{s_t^i, a_t^i\}_{i=0}^T\}$ by deploying the current policy $\pi_\theta(a_t \mid s_t)$.

2. Fit value function $V_\phi^\pi(s)$ by MC or TD estimation (update ϕ)

3. Compute action advantages $A^\pi(s_t^i, a_t^i) = G_t^{(i)} - V_\phi^\pi(s_t^i)$

4. $\nabla_\theta U(\theta) \approx \hat{g} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t^i \mid s_t^i) A^\pi(s_t^i, a_t^i)$

5. $\theta \leftarrow \theta + \alpha \nabla_\theta U(\theta)$

Sample returns are Q estimates!

Whether we use a baseline:

$$\hat{g} = \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\alpha_t^{(i)} \mid s_t^{(i)})\left(G_t^{(i)} - V_\phi^\pi(s_t^{(i)})\right)$$

or not:

$$\hat{g} = \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\alpha_t^{(i)} \mid s_t^{(i)})G_t^{(i)}$$

returns $G_t^{(i)} = \sum_{k=t}^{T} R(s_k^{(i)}, a_k^{(i)})$ are trying to estimate Q values from a single rollout:

$$Q^\pi(s,a) = \mathbb{E}[R_0 + R_1 + \cdots \mid S_0 = s, A_0 = a]$$

Sample returns are Q estimates!

Returns $G_t^{(i)} = \sum_{k=t}^{T} R(s_k^{(i)}, a_k^{(i)})$ are trying to estimate Q values from a single rollout:

$$Q^\pi(s,a) = \mathbb{E}[R_0 + R_1 + \cdots \mid S_0 = s, A_0 = a]$$

Minimize variance of such Q estimates by:

- discounting
- introducing a learnt approximation for the Q, as opposed to returns from single rollouts

By definition of the Q function

$$Q^{\pi,\gamma}(s,a) = \mathbb{E}[R_0 + \gamma R_1 + \gamma^2 R_2 \cdots \mid S_0 = s, A_0 = a]$$

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$$Q^{\pi,\gamma}(s,a) = \mathbb{E}[R_0 + \gamma R_1 + \gamma^2 R_2 \cdots \mid S_0 = s, A_0 = a]$$
$$= \mathbb{E}[R_0 + \gamma V^{\pi}(S_1) \mid S_0 = s, A_0 = a]$$

By definition of the Q function

$$
\begin{aligned}
Q^{\pi,\gamma}(s,a) &= \mathbb{E}[R_0 + \gamma R_1 + \gamma^2 R_2 \cdots \,|\, S_0 = s, A_0 = a] \\
&= \mathbb{E}[R_0 + \gamma V^{\pi}(S_1) \,|\, S_0 = s, A_0 = a] \\
&= \mathbb{E}[R_0 + \gamma R_1 + \gamma^2 V^{\pi}(S_2) \,|\, S_0 = s, A_0 = a]
\end{aligned}
$$

By definition of the Q function

$$
\begin{aligned}
Q^{\pi,\gamma}(s,a) &= \mathbb{E}[R_0 + \gamma R_1 + \gamma^2 R_2 \cdots \mid S_0 = s, A_0 = a] \\
&= \mathbb{E}[R_0 + \gamma V^{\pi}(S_1) \mid S_0 = s, A_0 = a] \\
&= \mathbb{E}[R_0 + \gamma R_1 + \gamma^2 V^{\pi}(S_2) \mid S_0 = s, A_0 = a] \\
&= \mathbb{E}[R_0 + \gamma R_1 + \gamma^2 R_2 + \gamma^3 V^{\pi}(S_3) \mid S_0 = s, A_0 = a] \\
&= \cdots
\end{aligned}
$$

If I have estimated $V^{\pi}_{\phi}(S)$, I can use it to estimate the Q values, I do not need a separate Q function approximate!

Advantage Actor-Critic

0. Initialize policy parameters θ and critic parameters ϕ .

1. Sample trajectories $\{\tau_i = \{s_t^i, a_t^i\}_{t=0}^T\}$ by deploying the current policy $\pi_\theta(a_t \mid s_t)$.

2. Fit value function $V_\phi^\pi(s)$ by MC or TD estimation (update ϕ)

3. Compute action advantages $A^\pi(s_t^i, a_t^i) = R(s_t^i, a_t^i) + \gamma V_\phi^\pi(s_{t+1}^i) - V_\phi^\pi(s_t^i)$

4. $\nabla_\theta U(\theta) \approx \hat{g} = \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\alpha_t^i \mid s_t^i)A^\pi(s_t^i, a_t^i)$

5. $\theta \leftarrow \theta + \alpha \nabla_\theta U(\theta)$

Advantage Actor-Critic

0. Initialize policy parameters θ and critic parameters ϕ .
1. Sample trajectories $\{\tau_i = \{s_t^i, a_t^i\}_{t=0}^{T}\}$ by deploying the current policy $\pi_\theta(a_t \mid s_t)$.
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5. $\theta \leftarrow \theta + \alpha \nabla_\theta U(\theta)$

How can we compute this in automatic differentiation packages, such us Tensor flow?

We need to write the expression that when differentiated will give that gradient.

$$\hat{U} = \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T} \log \pi_\theta(\alpha_t^i \mid s_t^i) A^\pi(s_t^i, a_t^i)$$

REINFORCE/Actor-critic training

- Stability of training neural networks requires the gradient updates to be de-correlated

- This is not the case if data arrives sequentially

- Gradient updates computed from some part of the space can cause the value (Q) function approximator to oscillate

- Our solution so far has been: Experience buffers where experience tuples are mixed and sampled from. Resulting sampled batches are more stationary that the ones encountered online (without buffer)

- This limits deep RL to off-policy methods, since data from an older policy are used to update the weights of the value approximator (critic) (except if we take care and weight such data under our current stochastic policy->importance sampling)

Asynchronous Deep RL for on policy learning

Asynchronous Methods for Deep Reinforcement Learning

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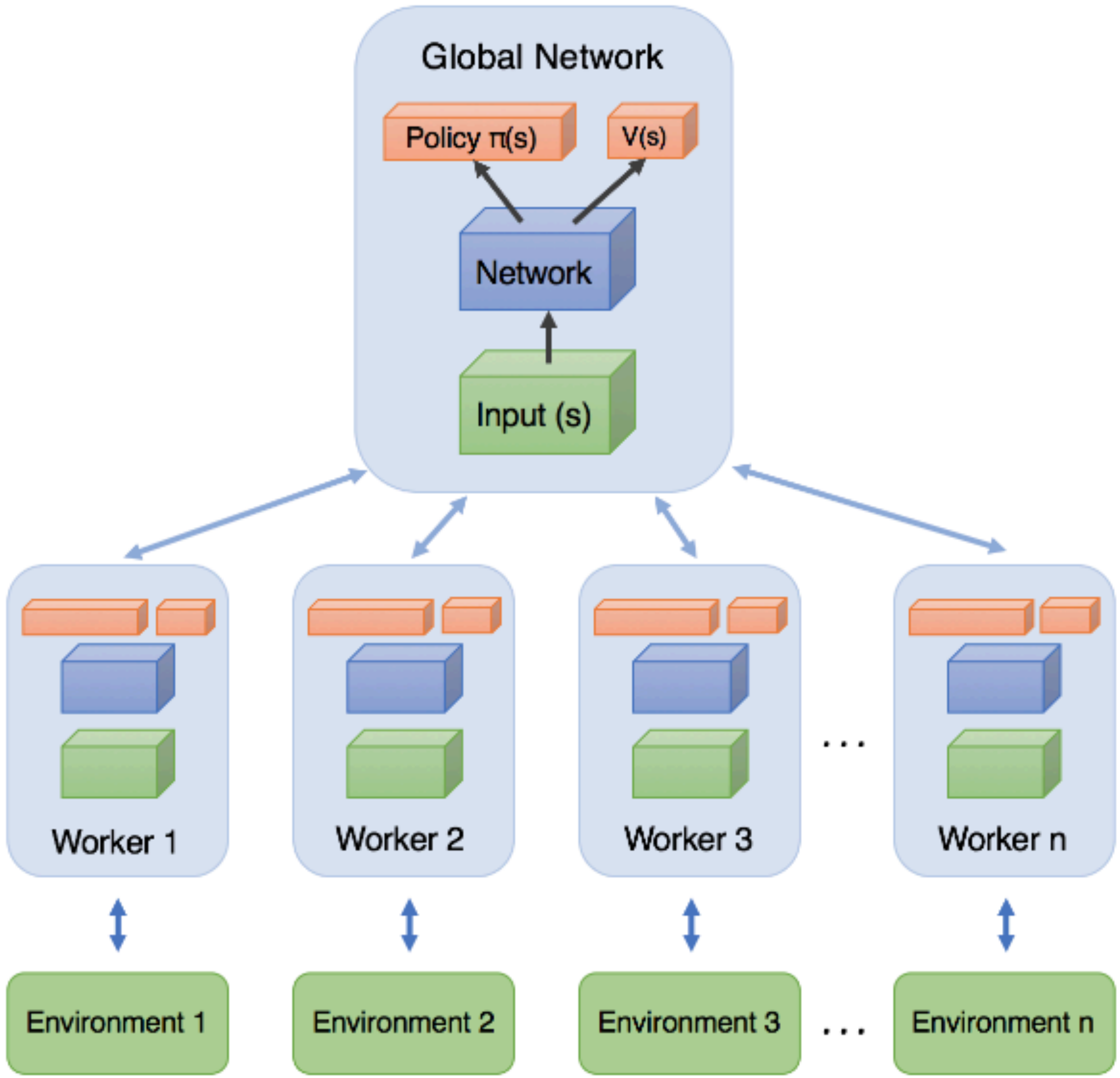
- Alternative: parallelize the collection of experience and stabilize training without experience buffers
- Multiple threads of experience, one per agent, each exploring in different part of the environment contributing experience tuples
- Different exploration strategies (e.g., various ϵ values) in different threads increase diversity
- Now you can train on-policy using any of our policy gradient methods

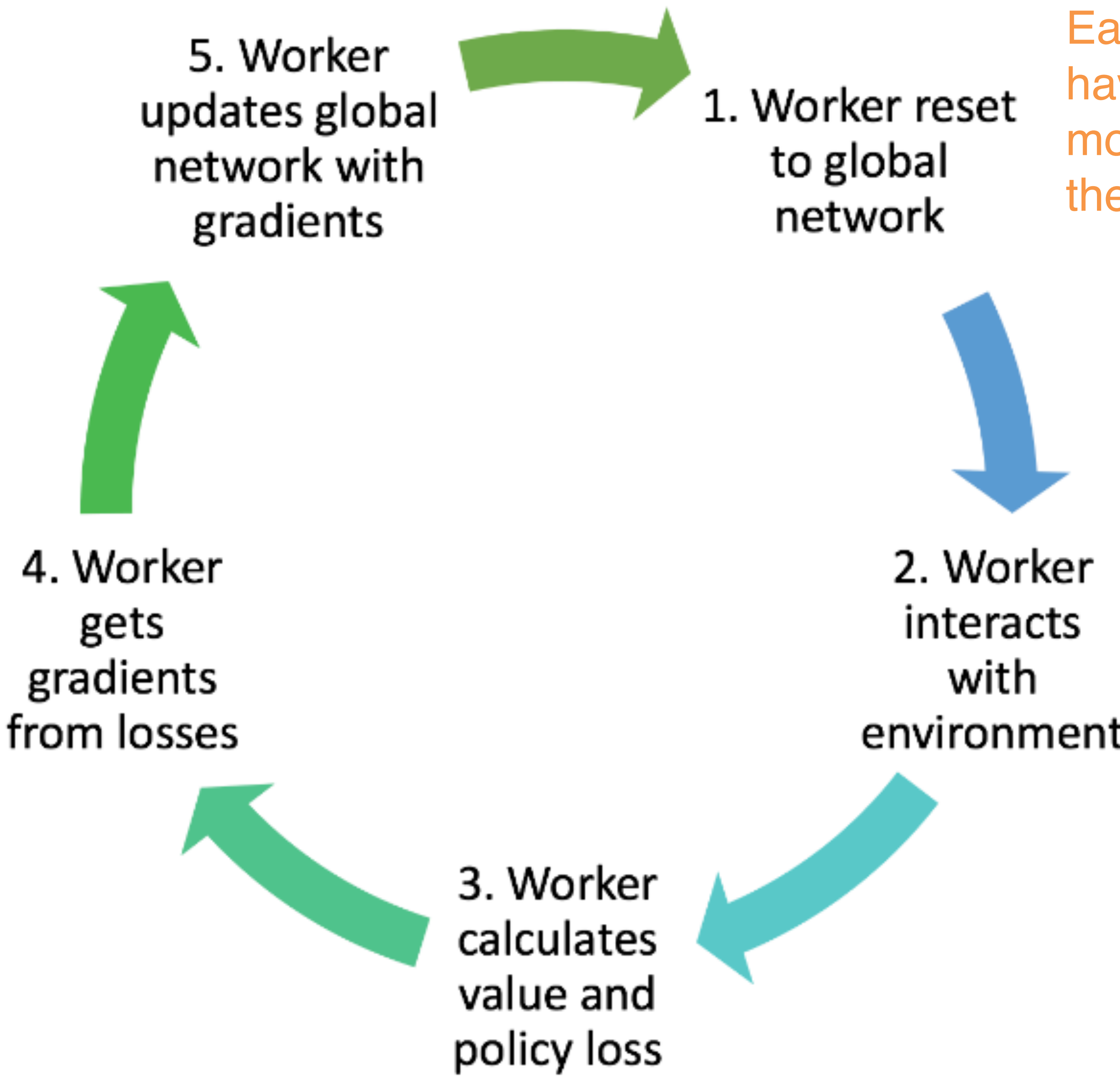
Distributed RL



Distributed Asynchronous RL-A3C

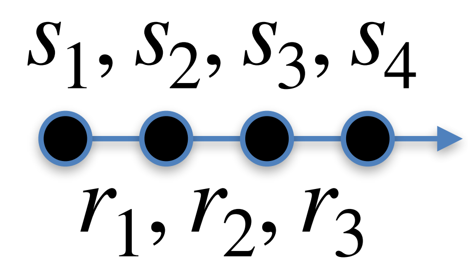
No locking

Each worker may have slightly modified version of the policy/critic



Algorithm S3 Asynchronous advantage actor-critic - pseudocode for each actor-learner thread.

// Assume global shared parameter vectors θ and θ_v and global shared counter $T = 0$
// Assume thread-specific parameter vectors θ' and θ'_v
Initialize thread step counter $t \leftarrow 1$
repeat
  Reset gradients: $d\theta \leftarrow 0$ and $d\theta_v \leftarrow 0$.
  Synchronize thread-specific parameters $\theta' = \theta$ and $\theta'_v = \theta_v$ — Copying the weights
  $t_{start} = t$
  Get state s_t
  repeat
    Perform a_t according to policy $\pi(a_t|s_t;\theta')$ — Rollout
    Receive reward r_t and new state s_{t+1}
    $t \leftarrow t + 1$
    $T \leftarrow T + 1$
  until terminal s_t **or** $t - t_{start} == t_{max}$
  $R = \begin{cases} 0 & \text{for terminal } s_t \\ V(s_t, \theta'_v) & \text{for non-terminal } s_t \text{// Bootstrap from last state} \end{cases}$
  for $i \in \{t-1, \dots, t_{start}\}$ **do**
    $R \leftarrow r_i + \gamma R$
    Accumulate gradients wrt θ': $d\theta \leftarrow d\theta + \nabla_{\theta'} \log \pi(a_i|s_i;\theta')(R - V(s_i;\theta'_v))$ — Advantage
    Accumulate gradients wrt θ'_v: $d\theta_v \leftarrow d\theta_v + \partial\left(R - V(s_i;\theta'_v)\right)^2 / \partial\theta'_v$ — Learning the critic
  end for
  Perform asynchronous update of θ using $d\theta$ and of θ_v using $d\theta_v$.
until $T > T_{max}$



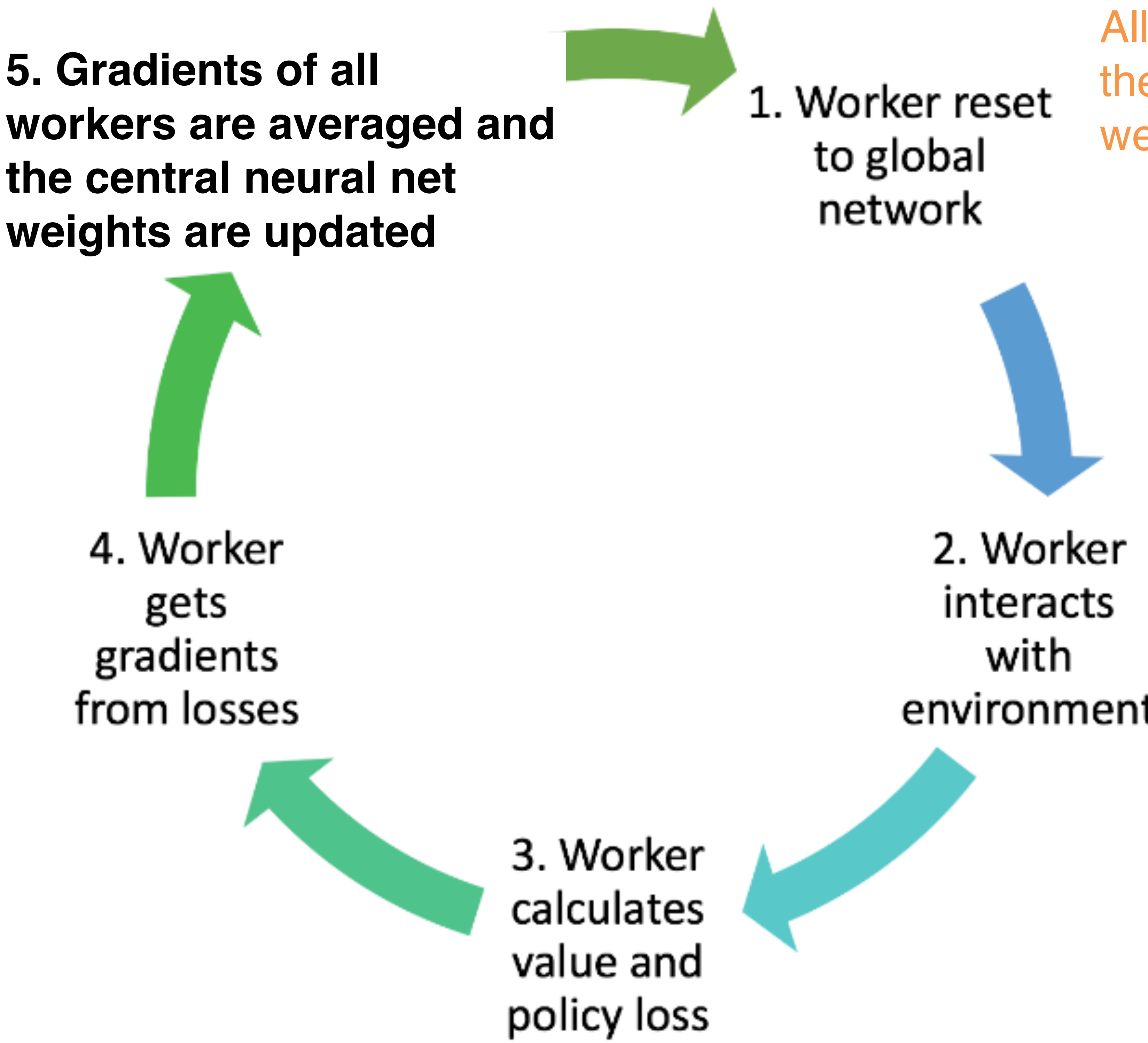
$R_3 - V(s_3)$

What is the approximation used for the advantage?

$R_3 = r_3 + \gamma V(s_4, \theta'_v)$ $A_3 = R_3 - V(s_3;\theta'_v)$

$R_2 = r_2 + \gamma r_3 + \gamma^2 V(s_4, \theta'_v)$ $A_2 = R_2 - V(s_2;\theta'_v)$

Distributed Synchronous RL-A2C



All worker may have the same actor/critic weights

Algorithm S3 Asynchronous advantage actor-critic - pseudocode for each actor-learner thread.

// Assume global shared parameter vectors θ and θ_v and global shared counter $T = 0$
// Assume thread-specific parameter vectors θ' and θ'_v
Initialize thread step counter $t \leftarrow 1$
repeat
Reset gradients: $d\theta \leftarrow 0$ and $d\theta_v \leftarrow 0$.

``We also found that adding the entropy of the policy π to the objective function improved exploration by discouraging premature convergence to suboptimal deterministic policies." So you need to add to the policy gradient: $+\beta \nabla_\theta \mathrm{H}(\pi_\theta(a_t | s_t; \theta))$

We will look into the entropy as part of the reward in later lecture

Perform asynchronous update of θ using $d\theta$ and of θ_v using $d\theta_v$.
until $T > T_{max}$

What is the approximation used for the advantage?

$R_3 = r_3 + \gamma V(s_4, \theta'_v)$ $\quad A_3 = R_3 - V(s_3; \theta'_v)$

$R_2 = r_2 + \gamma r_3 + \gamma^2 V(s_4, \theta'_v)$ $\quad A_2 = R_2 - V(s_2; \theta'_v)$

Actor-Critic

0. Initialize policy parameters θ and critic parameters ϕ .

1. Sample trajectories $\{\tau_i = \{s_t^i, a_t^i\}_{i=0}^T\}$ by deploying the current policy $\pi_\theta(a_t \mid s_t)$.

2. Fit value function $V_\phi^\pi(s)$ by MC or TD estimation (update ϕ)

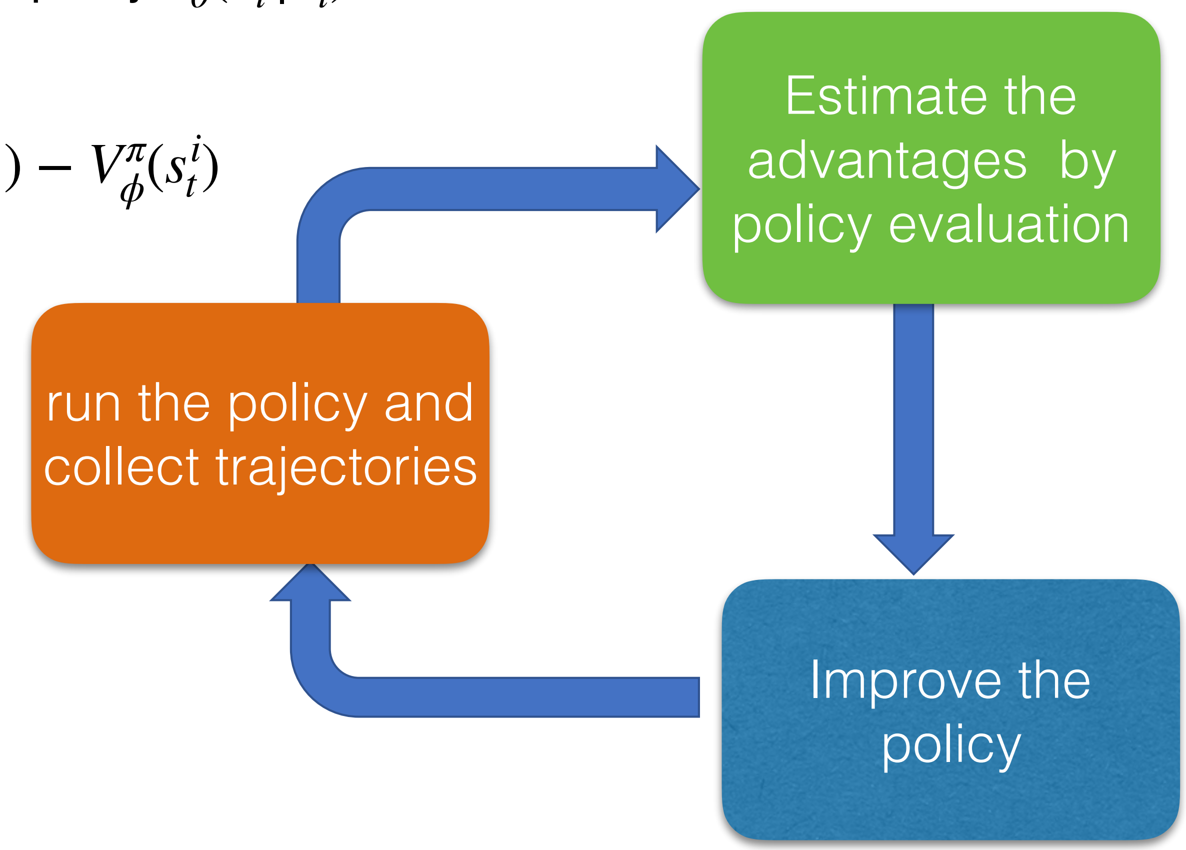
3. Compute action advantages $A^\pi(s_t^i, a_t^i) = R(s_t^i, a_t^i) + \gamma V_\phi^\pi(s_{t+1}^i) - V_\phi^\pi(s_t^i)$

4. $\nabla_\theta U(\theta) \approx \hat{g} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t^i \mid s_t^i) A^\pi(s_t^i, a_t^i)$

5. $\theta \leftarrow \theta + \alpha \nabla_\theta U(\theta)$

Actor-Critic as Policy Iteration

0. Initialize policy parameters θ and critic parameters ϕ .
1. Sample trajectories $\{\tau_i = \{s_t^i, a_t^i\}_{i=0}^T\}$ by deploying the current policy $\pi_\theta(a_t \mid s_t)$.
2. Fit value function $V_\phi^\pi(s)$ by MC or TD estimation (update ϕ)
3. Compute action advantages $A^\pi(s_t^i, a_t^i) = R(s_t^i, a_t^i) + \gamma V_\phi^\pi(s_{t+1}^i) - V_\phi^\pi(s_t^i)$
4. $\nabla_\theta U(\theta) \approx \hat{g} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t^i \mid s_t^i) A^\pi(s_t^i, a_t^i)$
5. $\theta \leftarrow \theta + \alpha \nabla_\theta U(\theta)$



Policy iteration:
1. Initialize policy π.
2. Deploy policy in the environment
3. Estimate and identify advantages A^π: actions whose $Q_\pi(s, a)$ value is higher than the state $V_\pi(s)$ value.
4. Change the policy to switch to those actions (or to make them more probable), resulting in π_{new}.
5. $\pi \rightarrow \pi_{new}$

Actor-Critic is on policy

0. Initialize policy parameters θ and critic parameters ϕ .

1. Sample trajectories $\{\tau_i = \{s_t^i, a_t^i\}_{i=0}^T\}$ by deploying the current policy $\pi_\theta(a_t \mid s_t)$.

2. Fit value function $V_\phi^\pi(s)$ by MC or TD estimation (update ϕ)

3. Compute action advantages $A^\pi(s_t^i, a_t^i) = R(s_t^i, a_t^i) + \gamma V_\phi^\pi(s_{t+1}^i) - V_\phi^\pi(s_t^i)$

4. $\nabla_\theta U(\theta) \approx \hat{g} = \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\alpha_t^i \mid s_t^i) A^\pi(s_t^i, a_t^i)$

5. $\theta \leftarrow \theta + \alpha \nabla_\theta U(\theta)$

The actor critic methods we have described are on policy methods: they use data of the current policy to improve the policy.

This lecture

When we have guarantees that we have policy improvement in AC methods?
What should be our stepwise in AC methods?
Can we use older data, data from previous versions of the policy, and train off-policy?

To guarantee improvement we need the policy not to change too much from iteration to iteration.

Can we use older data, data from previous versions of the policy?

Off-policy Policy Gradients with Importance Sampling

$$U(\theta) = \mathbb{E}_{\tau \sim \pi_\theta(\tau)} \left[R(\tau) \right]$$

$$= \sum_{\tau} \pi_\theta(\tau) R(\tau)$$

Off-policy Policy Gradients with Importance Sampling

$$
\begin{aligned}
U(\theta) &= \mathbb{E}_{\tau\sim\pi_\theta(\tau)}\left[R(\tau)\right] \\
&= \sum_\tau \pi_\theta(\tau)R(\tau) \\
&= \sum_\tau \pi_{\theta_{old}}(\tau)\frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)}R(\tau) \\
&= \mathbb{E}_{\tau\sim\pi_{\theta_{old}}}\frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)}R(\tau)
\end{aligned}
$$

Off-policy Policy Gradients with Importance Sampling

$$U(\theta) = \mathbb{E}_{\tau \sim \pi_\theta(\tau)} \left[R(\tau) \right]$$
$$= \sum_\tau \pi_\theta(\tau) R(\tau)$$
$$= \sum_\tau \pi_{\theta_{old}}(\tau) \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} R(\tau)$$
$$= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} R(\tau)$$

$$\nabla_\theta U(\theta) = \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\nabla_\theta \pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} R(\tau)$$
$$= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\pi_\theta(\tau)}{\pi_\theta(\tau)} \frac{\nabla_\theta \pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} R(\tau)$$
$$= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} \frac{\nabla_\theta \pi_\theta(\tau)}{\pi_\theta(\tau)} R(\tau)$$
$$= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} \nabla_\theta \log \pi_\theta(\tau) R(\tau)$$

Off-policy Policy Gradients with Importance Sampling

$$
\begin{aligned}
U(\theta) &= \mathbb{E}_{\tau \sim \pi_\theta(\tau)} \left[R(\tau) \right] \\
&= \sum_\tau \pi_\theta(\tau) R(\tau) \\
&= \sum_\tau \pi_{\theta_{old}}(\tau) \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} R(\tau) \\
&= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} R(\tau)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\theta U(\theta) &= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\nabla_\theta \pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} R(\tau) \\
&= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\pi_\theta(\tau)}{\pi_\theta(\tau)} \frac{\nabla_\theta \pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} R(\tau) \\
&= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} \frac{\nabla_\theta \pi_\theta(\tau)}{\pi_\theta(\tau)} R(\tau) \\
&= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} \nabla_\theta \log \pi_\theta(\tau) R(\tau)
\end{aligned}
$$

Gradient evaluated at θ_{old} is unchanged:

$$
\nabla_\theta U(\theta) \big|_{\theta=\theta_{old}} = \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \nabla_\theta \log \pi_\theta(\tau) \big|_{\theta=\theta_{old}} R(\tau)
$$

Off-policy Policy Gradients with Importance Sampling

$$\nabla_\theta U(\theta) = \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} \nabla_\theta \log \pi_\theta(\tau) R(\tau)$$

$$\pi_\theta(\tau) = p(s_0) \prod_{t=1}^{T} p(s_{t+1} \mid s_t, a_t) \cdot \pi_\theta(a_t \mid s_t)$$

$\Rightarrow$

$$\frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} = \frac{p(s_0) \prod_{t=1}^{T} p(s_{t+1} \mid s_t, a_t) \cdot \pi_\theta(a_t \mid s_t)}{p(s_0) \prod_{t=1}^{T} p(s_{t+1} \mid s_t, a_t) \cdot \pi_{\theta_{old}}(a_t \mid s_t)} = \prod_{t=1}^{T} \frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)}$$

$$\begin{aligned} \nabla_\theta U(\theta) &= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \prod_{t=1}^{T} \frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t \mid s_t) \sum_{t=1}^{T} R(s_t, a_t) \\ &= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \sum_{t=1}^{T} \left[\nabla_\theta \log \pi_\theta(\alpha_t \mid s_t) \left(\prod_{t=1}^{T} \frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)} \right) \left(\sum_{t=1}^{T} R(s_t, a_t) \right) \right] \end{aligned}$$

Off-policy Policy Gradients with Importance Sampling

$$\nabla_\theta U(\theta) = \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} \nabla_\theta \log \pi_\theta(\tau) R(\tau)$$

$$\pi_\theta(\tau) = p(s_0) \prod_{t=1}^{T} p(s_{t+1} \mid s_t, a_t) \cdot \pi_\theta(a_t \mid s_t) \quad \Rightarrow \quad \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} = \frac{p(s_0) \prod_{t=1}^{T} p(s_{t+1} \mid s_t, a_t) \cdot \pi_\theta(a_t \mid s_t)}{p(s_0) \prod_{t=1}^{T} p(s_{t+1} \mid s_t, a_t) \cdot \pi_{\theta_{old}}(a_t \mid s_t)} = \prod_{t=1}^{T} \frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)}$$

$$\begin{aligned}
\nabla_\theta U(\theta) &= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \prod_{t=1}^{T} \frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t \mid s_t) \sum_{t=1}^{T} R(s_t, a_t) \\
&= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \sum_{t=1}^{T} \left[\nabla_\theta \log \pi_\theta(\alpha_t \mid s_t) \left(\prod_{t=1}^{T} \frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)} \right) \left(\sum_{t=1}^{T} R(s_t, a_t) \right) \right]
\end{aligned}$$

Causal temporal structure?

$$\nabla_\theta U(\theta) = \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \sum_{t=1}^{T} \left[\nabla_\theta \log \pi_\theta(\alpha_t \mid s_t) \left(\prod_{t'=1}^{t} \frac{\pi_\theta(a_{t'} \mid s_{t'})}{\pi_{\theta_{old}}(a_{t'} \mid s_{t'})} \right) \left(\sum_{t'=t}^{T} R(s_{t'}, a_{t'}) \right) \right]$$

Off-policy Policy Gradients with Importance Sampling

$$\nabla_\theta U(\theta) = \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} \nabla_\theta \log \pi_\theta(\tau) R(\tau)$$

$$\pi_\theta(\tau) = p(s_0) \prod_{t=1}^{T} p(s_{t+1} | s_t, a_t) \cdot \pi_\theta(a_t | s_t)$$

$$\Rightarrow \quad \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} = \frac{p(s_0) \prod_{t=1}^{T} p(s_{t+1} | s_t, a_t) \cdot \pi_\theta(a_t | s_t)}{p(s_0) \prod_{t=1}^{T} p(s_{t+1} | s_t, a_t) \cdot \pi_{\theta_{old}}(a_t | s_t)} = \prod_{t=1}^{T} \frac{\pi_\theta(a_t | s_t)}{\pi_{\theta_{old}}(a_t | s_t)}$$

$$\nabla_\theta U(\theta) = \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \prod_{t=1}^{T} \frac{\pi_\theta(a_t | s_t)}{\pi_{\theta_{old}}(a_t | s_t)} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t | s_t) \sum_{t=1}^{T} R(s_t, a_t)$$

$$= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \sum_{t=1}^{T} \left[\nabla_\theta \log \pi_\theta(\alpha_t | s_t) \left(\prod_{t=1}^{T} \frac{\pi_\theta(a_t | s_t)}{\pi_{\theta_{old}}(a_t | s_t)} \right) \left(\sum_{t=1}^{T} R(s_t, a_t) \right) \right]$$

These terms can explode or vanish! Not very useful due to the large variance they cause in the estimator.

Causal temporal structure?

$$\nabla_\theta U(\theta) = \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \sum_{t=1}^{T} \left[\nabla_\theta \log \pi_\theta(\alpha_t | s_t) \left(\prod_{t'=1}^{t} \frac{\pi_\theta(a_{t'} | s_{t'})}{\pi_{\theta_{old}}(a_{t'} | s_{t'})} \right) \left(\sum_{t'=t}^{T} R(s_{t'}, a_{t'}) \right) \right]$$

Approximate Off-policy Policy Gradients with Importance Sampling

$$U(\theta) = \mathbb{E}_{\tau \sim \pi_\theta(\tau)} \left[R(\tau) \right]$$

Let's write the objective w.r.t. expectations for each time step using per-timestep state marginal distributions:

$$U(\theta) = \sum_{t=1}^{T} \mathbb{E}_{(s_t, a_t) \sim p_\theta(s_t, a_t)} R(s_t, a_t) = \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_\theta(s_t)} \left[\mathbb{E}_{a_t \sim \pi_\theta(a_t | s_t)} R(s_t, a_t) \right]$$

We define the state marginal distribution as the probability that the policy visits state s:

$$p_\theta(s) = \mathbb{E}_{s_1 \sim p_0(s), a \sim \pi(\cdot|s), s_{t+1} \sim p(S|s_t, a_t)} \frac{1}{T} \sum_{t=1}^{T} \mathbf{1}(s_t = s)$$

Approximate Off-policy Policy Gradients with Importance Sampling

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Now let's use importance sampling for state and action sampling:

$$U^{IS}(\theta) = \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_{\theta_{old}}(s_t)} \frac{p_\theta(s_t)}{p_{\theta_{old}}(s_t)} \left[\mathbb{E}_{a_t \sim \pi_{\theta_{old}}(a_t | s_t)} \frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)} R(s_t, a_t) \right]$$

Approximate Off-policy Policy Gradients with Importance Sampling

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Do we know those quantities?

Approximate Off-policy Policy Gradients with Importance Sampling

$$U(\theta) = \mathbb{E}_{\tau \sim \pi_\theta(\tau)} \left[R(\tau) \right]$$

Let's write the objective w.r.t. expectations for each time step using per-timestep state marginal distributions:

$$U(\theta) = \sum_{t=1}^{T} \mathbb{E}_{(s_t,a_t) \sim p_\theta(s_t,a_t)} R(s_t, a_t) = \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_\theta(s_t)} \left[\mathbb{E}_{a_t \sim \pi_\theta(a_t|s_t)} R(s_t, a_t) \right]$$

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Approximation!

$$U^{IS}(\theta) = \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_{\theta_{old}}(s_t)} \cancel{\frac{p_\theta(s_t)}{p_{\theta_{old}}(s_t)}} \left[\mathbb{E}_{a_t \sim \pi_{\theta_{old}}(a_t|s_t)} \frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)} R(s_t, a_t) \right]$$

$$U^{IS}(\theta) \approx \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_{\theta_{old}}(s_t)} \left[\mathbb{E}_{a_t \sim \pi_{\theta_{old}}(a_t|s_t)} \frac{\pi_\theta(a_t)}{\pi_{\theta_{old}}(a_t)} R(s_t, a_t) \right]$$

Approximate Off-policy Policy Gradients with Importance Sampling

$$U(\theta) = \mathbb{E}_{\tau \sim \pi_\theta(\tau)} \left[R(\tau) \right]$$

Let's write the objective w.r.t. expectations for each time step using per-timestep state marginal distributions:

$$U(\theta) = \sum_{t=1}^{T} \mathbb{E}_{(s_t,a_t) \sim p_\theta(s_t,a_t)} R(s_t, a_t) = \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_\theta(s_t)} \left[\mathbb{E}_{a_t \sim \pi_\theta(a_t|s_t)} R(s_t, a_t) \right]$$

Now let's use importance sampling for state and action sampling:

$$U^{IS}(\theta) = \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_{\theta_{old}}(s_t)} \frac{p_\theta(s_t)}{p_{\theta_{old}}(s_t)} \left[\mathbb{E}_{a_t \sim \pi_{\theta_{old}}(a_t|s_t)} \frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)} R(s_t, a_t) \right]$$

Approximation!

$$U^{IS}(\theta) = \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_{\theta_{old}}(s_t)} \cancel{\frac{p_\theta(s_t)}{p_{\theta_{old}}(s_t)}} \left[\mathbb{E}_{a_t \sim \pi_{\theta_{old}}(a_t|s_t)} \frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)} R(s_t, a_t) \right]$$

Do you know how to optimize this objective?

$$U^{IS}(\theta) \approx \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_{\theta_{old}}(s_t)} \left[\mathbb{E}_{a_t \sim \pi_{\theta_{old}}(a_t|s_t)} \frac{\pi_\theta(a_t)}{\pi_{\theta_{old}}(a_t)} R(s_t, a_t) \right]$$

Approximate Off-policy Policy Gradients with Importance Sampling

$$U(\theta) = \mathbb{E}_{\tau \sim \pi_\theta(\tau)} \left[R(\tau) \right]$$

Let's write the objective w.r.t. expectations for each time step using per-timestep state marginal distributions:

$$U(\theta) = \sum_{t=1}^{T} \mathbb{E}_{(s_t,a_t)\sim p_\theta(s_t,a_t)} R(s_t, a_t) = \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_\theta(s_t)} \left[\mathbb{E}_{a_t \sim \pi_\theta(a_t|s_t)} R(s_t, a_t) \right]$$

Now let's use importance sampling for state and action sampling:

$$U^{IS}(\theta) = \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_{\theta_{old}}(s_t)} \frac{p_\theta(s_t)}{p_{\theta_{old}}(s_t)} \left[\mathbb{E}_{a_t \sim \pi_{\theta_{old}}(a_t|s_t)} \frac{\pi_\theta(a_t | s_t)}{\pi_{\theta_{old}}(a_t | s_t)} R(s_t, a_t) \right]$$

Approximation!

$$U^{IS}(\theta) = \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_{\theta_{old}}(s_t)} \frac{p_\theta(s_t)}{p_{\theta_{old}}(s_t)} \left[\mathbb{E}_{a_t \sim \pi_{\theta_{old}}(a_t|s_t)} \frac{\pi_\theta(a_t | s_t)}{\pi_{\theta_{old}}(a_t | s_t)} R(s_t, a_t) \right]$$

Do you know how to optimize this objective? Yes, we will sample trajectories from π_{old} and use likelihood ratio estimator.

$$U^{IS}(\theta) \approx \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_{\theta_{old}}(s_t)} \left[\mathbb{E}_{a_t \sim \pi_{\theta_{old}}(a_t|s_t)} \frac{\pi_\theta(a_t | s_t)}{\pi_{\theta_{old}}(a_t | s_t)} R(s_t, a_t) \right]$$

Approximate Off-policy Policy Gradients with Importance Sampling

Do you know how to optimize this objective? Yes, we will sample trajectories from π_{old} and use likelihood ratio estimator.

$$\hat{U}^{IS}(\theta) = \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_{\theta_{old}}(s_t)} \left[\mathbb{E}_{a_t \sim \pi_{\theta_{old}}(a_t|s_t)} \frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)} R(s_t, a_t) \right]$$

$$\nabla_\theta \hat{U}^{IS}(\theta) = \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \sum_{t=1}^{T} \left[\nabla_\theta \log \pi_\theta(\alpha_t \mid s_t) \left(\frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)} \right) \left(\sum_{t'=t}^{T} R(s_{t'}, a_{t'}) \right) \right]$$

Compare to the exact IS gradient:

$$\nabla_\theta U(\theta) = \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \sum_{t=1}^{T} \left[\nabla_\theta \log \pi_\theta(\alpha_t \mid s_t) \left(\prod_{t'=1}^{t} \frac{\pi_\theta(a_{t'} \mid s_{t'})}{\pi_{\theta_{old}}(a_{t'} \mid s_{t'})} \right) \left(\sum_{t'=t}^{T} R(s_{t'}, a_{t'}) \right) \right]$$