Distributional RL

Overview

• Motivation
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• Distributional Bellman Equations & Operators
• C51 Algorithm
• Current Research Topics
Distributional RL

Motivation

How long do you have to wait for the bus?

Frequency

Minutes

DIRECT ANALYSIS OF THE PERCEIVED IMPORTANCE OF ATTRIBUTES OF RELIABILITY OF TRAVEL MODES IN URBAN TRAVEL

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ABSTRACT

Reliability of travel modes was found to be the most important characteristic of transportation systems in several attitudinal investigations of individual travel behavior. This paper represents the first part of a research effort aimed at gaining a better understanding of the characteristics of reliability of transportation modes in urban travel. In this research, reliability characteristics are identified; their importance relative to each other is assessed, and an insight into possible structure of an objective reliability index is discussed. The research is based on perceived values of reliability, which were identified through a large attitudinal survey conducted in the Chicago metropolitan area.
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Motivation

• So far we have been focusing on maximising Expected returns

• Distributional versions of Bellman equations have been studied for almost as long as the usual expected versions, originally motivated by risk-aware decision making settings

• Recently (2017), researchers have found that taking a distributional approach to DRL has benefits even if all we care about is expected return

• Still being studied to gain understanding as to why distributional approach helps the expected setting
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Preliminaries

- What is an operator?
- What does it mean for two discrete random variables to be equivalent?
- What is the difference between $\mathcal{F}$ and $\mathcal{L}$?
- What is the difference between $z(s, a)$ and $z(S, A)$?
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Preliminaries

- What is an operator?
  - Suppose we have a space of functions $\mathcal{F}$, where each $f \in \mathcal{F}$ is a function of the form $f : \mathcal{X} \rightarrow \mathcal{Y}$
  - We call $T$ an operator if it is a mapping of the form $T : \mathcal{F} \rightarrow \mathcal{F}$
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Preliminaries

- What is an operator?
  - We write:
    - $f \in \mathcal{F}$ to denote the function
    - $f(x)$ to denote the function evaluated at point $x \in \mathcal{X}$
    - $Tf$ to denote the function returned by the operator $T$
    - $(Tf)(x)$ to denote the function returned by the operator $T$ evaluated at point $x \in \mathcal{X}$
• **What is an operator?**

  • **Example:** if \( \mathcal{F} \) is the space of functions of the form \( \mathcal{S} \times \mathcal{A} \to \mathbb{R} \), then the Bellman operator \( T^\pi : \mathcal{F} \to \mathcal{F} \) is an operator on this space of functions, and for any \( f \in \mathcal{F} \), we know that \((T^\pi f)(s, a)\) will be closer to \( q^\pi(s, a) \) than \( f(s, a) \) for any \((s, a) \in \mathcal{S} \times \mathcal{A} \) (because \( T^\pi \) is a contraction)

  • In the rest of these slides, \( \mathcal{F} \) will denote the space functions of the form \( \mathcal{S} \times \mathcal{A} \to \mathbb{R} \)
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Preliminaries

• What does it mean for two discrete random variables to be equivalent?

  • Suppose $X \sim P_X$ is a discrete RV with pmf $p_X(x)$
  • Suppose $Y \sim P_Y$ is a discrete RV with pmf $p_Y(y)$
  • We say that $X$ is equivalent to $Y$, and write $X \overset{D}{=} Y$ if $p_X(x) = p_Y(y)$ whenever $x = y$
  • Note that this says nothing about the dependency between $X$ and $Y$
  • (We also write $P_X \overset{D}{=} P_Y$, which can be interpreted as implying that $X \overset{D}{=} Y$)
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Preliminaries

- **What is the difference between $\mathcal{F}$ and $\mathcal{Z}$?**

  - $\mathcal{F}$ is space of functions where each function $f \in \mathcal{F}$ is a mapping of the form $f : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$

  - $\mathcal{Z}$ is a special space of functions used in Distributional RL

  - Each function $z \in \mathcal{Z}$ is a mapping of the form $z : \mathcal{S} \times \mathcal{A} \to \Delta(\mathbb{R})$

  - Here I am using $\Delta(\mathbb{R})$ to denote the set of discrete probability distributions over the real numbers
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Preliminaries

• What is the difference between $\mathcal{F}$ and $\mathcal{L}$?

• Example: $z_1 \in \mathcal{L}$ could be defined as

  $z_1(s, a) = \text{Uniform}([1,2,...,10])$ for all $(s, a) \in \mathcal{S} \times \mathcal{A}$

• Example: $z_2 \in \mathcal{L}$ could be defined as

  $z_2(s, a) = \begin{cases} 
  \text{Uniform}([1,2,...,10]) & \text{if } s > 3 \\
  \text{Binomial}(10,0.5) & \text{otherwise}
  \end{cases}$ for all $(s, a) \in \mathcal{S} \times \mathcal{A}$
Distributional RL
Preliminaries

• What is the difference between $\mathcal{F}$ and $\mathcal{E}$?

• Example: $z_R \in \mathcal{E}$ could be defined as

$$z_R(s, a) = \mathbb{P} \left( R \in \cdot \mid S = s, A = a \right) \text{ for all } (s, a) \in \mathcal{S} \times \mathcal{A}$$
• What is the difference between $z(s, a)$ and $z(S, A)$?

- $z \in \mathcal{F}$ is a deterministic function of the form $z : \mathcal{S} \times \mathcal{A} \rightarrow \Delta(\mathbb{R})$
- $(s, a) \in \mathcal{S} \times \mathcal{A}$ is an arbitrary (i.e. not random) point in state-action space
- $z(s, a)$ lies in $\Delta(\mathbb{R})$, so it is a probability distribution, but it is deterministic in the sense that it is a specific probability distribution in the set $\Delta(\mathbb{R})$. 
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Preliminaries

- What is the difference between \( z(s, a) \) and \( z(S, A) \)?

  - \( S, A \) are two random variables

  - Ordinarily, \( z(S, A) \) would be random, and would therefore have a probability distribution defined over \( \Delta(\mathbb{R}) \) (i.e. a probability distribution over probability distributions). However, it is common in Distributional RL to use \( z(S, A) \) to denote the mixture distribution suggested by the RVs \( S, A \), which is not random.
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Preliminaries

• What is the difference between $z(s, a)$ and $z(S, A)$?

• Example:

  Suppose $z_R(s, a) = \mathbb{P} \left( R \in \cdot \mid S = s, A = a \right)$ for all $(s, a) \in \mathcal{S} \times \mathcal{A}$
  
  Suppose $S, A = \begin{cases} s_1, a_1 & \text{with probability 0.5} \\
  s_2, a_2 & \text{with probability 0.5} \end{cases}$

  Then $z_R(S, A)$ denotes the mixture distribution combining $z_R(s_1, a_1)$ and $z_R(s_2, a_2)$ with equal weights (so we have $z_R(S, A) \in \Delta(\mathbb{R})$).
Questions?

• What is an operator?

• What does it mean for two discrete random variables to be equivalent?

• What is the difference between $\mathcal{F}$ and $\mathcal{E}$?

• What is the difference between $z(s, a)$ and $z(S, A)$?
Quick note:

There is one point where we revert to the usual stochastic interpretation for $z(S, A)$ instead of its mixture distribution interpretation. I will point this out when it happens.
Distributional RL

Setting

- Finite, episodic MDP $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{R}, p, \rho_0, \gamma)$
- Finitely many realisable discounted returns
- Assume that $(R \independent S' | S, A)$ i.e. $p(r, s' | s, a) = p_R(r | s, a)p_{S'}(s' | s, a)$
- Stochastic policy $\pi : \mathcal{S} \times \mathcal{A} \rightarrow [0,1]$
- $q^\pi$ is the true state-action value function for policy $\pi$
- $q$ is an arbitrary function of the form $q : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$ (so we have $q \in \mathcal{F}$)
Distributional RL

Setting

- Bellman Equation:
  \[ q^\pi(s, a) = \mathbb{E} \left[ R + \gamma q^\pi(S', A') \middle| S = s, A = a \right] \]

- Bellman Operator \( T^\pi \) applied to \( q \) and evaluated at \((s, a)\):
  \[ (T^\pi q)(s, a) = \mathbb{E} \left[ R + \gamma q(S', A') \middle| S = s, A = a \right] \]

- Note that \( T^\pi : \mathcal{F} \rightarrow \mathcal{F} \)
In Distributional RL, we have something similar, but we need to be careful with our interpretation.

- $z^\pi \in \mathcal{Z}$ is the true distributional state-action value function for policy $\pi$.

  - Remember that $z^\pi : \mathcal{S} \times \mathcal{A} \to \Delta(\mathbb{R})$.

  - $z^\pi(s, a)$ is the discrete probability distribution.

  - $\mathbb{P}\left(\sum_{t=0}^{\infty} \gamma^t R_{t+1} \in \cdot \mid S_0 = s, A_0 = a\right)$ assuming policy $\pi$ in MDP $\mathcal{M}$.

  - Note that (with abuse of notation) $\mathbb{E} \left[z^\pi(s, a)\right] = q^\pi(s, a)$ and $\mathbb{E} \left[z^\pi\right] = q^\pi$.
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Distributional Bellman Equations & Operators

- Distributional Bellman Equation:
  
  \[
  z^\pi(s, a) \overset{D}{=} z_R(s, a) + \gamma z^\pi(S', A')
  \]

  - \(z_R(s, a)\) is the discrete probability distribution \(\mathbb{P}(R \in \cdot \mid S = s, A = a)\).
  
  - \(z_R(s, a) + \gamma z^\pi(S', A')\) is abusive notation for the discrete probability distribution of the RV \(X + \gamma Y\) where \(X \sim z_R(s, a)\) and \(Y \sim z^\pi(S', A')\) with \(S', A'\) distributed according to MDP \(\mathcal{M}\) under policy \(\pi\) from state-action pair \((s, a)\).

  - Remember that \(z^\pi(S', A')\) denotes the mixture distribution!
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Distributional Bellman Equations & Operators

- \( z^\pi(s, a) \overset{D}{=} z_R(s, a) + \gamma z^\pi(S', A') \)

\[ S', A' = \begin{cases} 
    s_1, a_1 & \text{with probability 0.5} \\
    s_2, a_2 & \text{with probability 0.5}
\end{cases} \]
Distributional RL
Distributional Bellman Equations & Operators

• $z^\pi(s, a) \overset{D}{=} z_R(s, a) + \gamma z^\pi(S', A')$

$S', A' = \begin{cases} 
    s_1, a_1 & \text{with probability 0.5} \\
    s_2, a_2 & \text{with probability 0.5}
\end{cases}$

$\mathbb{P}\left( \sum_{t=0}^{\infty} \gamma^t R_{t+1} \in \cdot \mid S_0 = s_1, A_0 = a_1 \right)$
Distributional RL
Distributional Bellman Equations & Operators

- \( z^\pi(s, a) \overset{D}{=} z_R(s, a) + \gamma z^\pi(S', A') \)

\( S', A' = \begin{cases} 
  s_1, a_1 & \text{with probability 0.5} \\
  s_2, a_2 & \text{with probability 0.5}
\end{cases} \)

\[ \mathbb{P}\left( \sum_{t=0}^{\infty} \gamma^t R_{t+1} \in \cdot \mid S_0 = s_2, A_0 = a_2 \right) \]
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Distributional Bellman Equations & Operators

- \( z^\pi(s, a) \overset{D}{=} z_R(s, a) + \gamma z^\pi(S', A') \)

\( S', A' = \begin{cases} 
    s_1, a_1 & \text{with probability 0.5} \\
    s_2, a_2 & \text{with probability 0.5}
\end{cases} \)

\[
\frac{1}{2} \mathbb{P} \left( \sum_{t=0}^{\infty} \gamma^t R_{t+1} \in \cdot \mid S_0 = s_1, A_0 = a_1 \right) + \\
\frac{1}{2} \mathbb{P} \left( \sum_{t=0}^{\infty} \gamma^t R_{t+1} \in \cdot \mid S_0 = s_2, A_0 = a_2 \right)
\]
Distributional RL
Distributional Bellman Equations & Operators

- $z^\pi(s, a) \overset{D}{=} z_R(s, a) + \gamma z^\pi(S', A')$
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Distributional Bellman Equations & Operators

- \( z^\pi(s, a) \overset{D}{=} z_R(s, a) + \gamma z^\pi(S', A') \)
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Distributional Bellman Equations & Operators

\[ z^\pi(s, a) \overset{D}{=} z_R(s, a) + \gamma z^\pi(S', A') \]

+ \text{Distributional Bellman Equations}
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Distributional Bellman Equations & Operators

1. \( z^\pi(s, a) \overset{D}{=} z_R(s, a) + \gamma z^\pi(S', A') \)

2. In general, to find the pmf of the sum of two (conditionally) independent RVs, we need to *convolve* their individual (conditional) pmfs.

3. When one of the RVs is a point mass, convolution becomes a shift.
Distributional RL
Distributional Bellman Equations & Operators

- \( z^\pi(s, a) \overset{D}{=} z_R(s, a) + \gamma z^\pi(S', A') \)

- RHS has 3 steps:
  - find mixture \( z(S', A') \)
  - scale by \( \gamma \)
  - convolve/shift using \( z_R(s, a) \)
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Distributional Bellman Equations & Operators

Just like in classical Dynamic Programming, we can define a Distributional Bellman Operator $U^\pi$. Specifically, the Distributional Bellman Operator applied to $z$ and evaluated at $(s, a)$ is given by:

$$(U^\pi z)(s, a) \overset{D}{=} z_R(s, a) + \gamma z(S', A')$$

That is, we start with some arbitrary $z \in \mathcal{Z}$, and we update the distribution at each $(s, a)$ to be the distribution of the RV $X + \gamma Y$, where $X \sim z_R(s, a)$, and $Y \sim z(S', A')$.

Note that $U^\pi : \mathcal{Z} \rightarrow \mathcal{Z}$, whereas $T^\pi : \mathcal{F} \rightarrow \mathcal{F}$.
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Distributional Bellman Equations & Operators

• In [Bellemare et al. 2017], they showed that $U^\pi$ is a contraction, similar to how $T^\pi$ is a contraction

• For any two $q, q' \in \mathcal{F}$, we needed to find the biggest difference across $\mathcal{S} \times \mathcal{A}$, and “difference” was trivial to calculate: just $|q(s, a) - q'(s, a)|$

• For any two $z, z' \in \mathcal{E}$, now it is less straightforward to measure “difference” between $z(s, a)$ and $z'(s, a)$

• Turns out Wasserstein metric is a convenient way to measure distance for analysing $U^\pi$
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Distributional Bellman Equations & Operators

• Intuitively:
  • **KL divergence** only penalises when two distributions disagree on the probability of an event, but there is no notion of “similarity” of events.
  • **Wasserstein** distances additionally measure similarity of events and penalise disagreements accordingly.
Questions?

- $z^\pi(s, a) \overset{D}{=} z_R(s, a) + \gamma z^\pi(S', A')$
- find mixture $z(S', A')$
- scale by $\gamma$
- convolve/shift using $z_R(s, a)$
Unfortunately, things are not so simple for the control case.

Main issue is that it’s not clear what is the best way to generalise the maximisation step in the Bellman Optimality Equations: how should we maximise over distributions?

\[
(U^*z)(s, a) \overset{D}{=} z_R(s, a) + \gamma \max_{a'} z(S', a')
\]

not defined
In [Bellemare et al. ’17], they propose the definition $U^* z := U^{\pi^*} z$, where $\pi^*$ is any greedy policy w.r.t. $\mathbb{E}[z]$.

Unfortunately, $U^*$ is not a contraction in any metric. However, the authors show that $\lim_{k \to \infty} \mathbb{E}[(U^*)^k z] \to q^*$ which motivates their proposed algorithm.
For a tractable algorithm, we need a differentiable space of functions $\mathcal{E}_\theta$ that interacts well with the Distributional Bellman (Optimality) Operators.

In [Bellemare et al. ’17], authors propose using a categorical distribution parameterised by $\theta$ and then modify DQN to train approximate distributional value functions.

They have a minimum and maximum return, and then they discretise the real number line into $N$ uniform intervals (they found $N = 51$ to be effective for playing Atari, hence the name).
C51 Algorithm

- $C51 = DQN$, but $z_\theta$ instead of $q_\theta$; train $z_\theta$ by taking stochastic gradient descent steps on loss:
  
  $KL \left( \delta(R) + \gamma z_\bar{\theta}(S', A^*) \bigg\| z_\theta(S, A) \right)$

  - with $(S, A, R, S')$ from replay buffer and $A^* = \arg \max_{a'} \mathbb{E} [z_\bar{\theta}(S', a')]$
**Distributional RL**

**C51 Algorithm**

- C51 = DQN, but $z_\theta$ instead of $q_\theta$; train $z_\theta$ by taking stochastic gradient descent steps on loss:
  
  \[ KL \left( \delta(R) + \gamma z_\theta(S', A^\star) \right\| z_\theta(S, A) \right) \]

  - with $(S, A, R, S')$ from replay buffer and $A^\star = \arg \max_{a'} \mathbb{E} \left[ z_\theta(S', a') \right]$

  - Note: we are returning to the more standard *stochastic* interpretation of $z(S, A)$ here instead of the mixture interpretation during operator analysis

  - i.e. $z_\theta(S, A)$, $z_\theta(S', A^\star)$ are random *not* mixture distributions
C51 Algorithm

- C51 = DQN, but $z_\theta$ instead of $q_\theta$; train $z_\theta$ by taking stochastic gradient descent steps on loss:

$$KL \left( \delta(R) + \gamma z_{\bar{\theta}}(S', A^*) \middle\| z_\theta(S, A) \right)$$

- with $(S, A, R, S')$ from replay buffer and $A^* = \arg\max_a \mathbb{E} \left[ z_{\bar{\theta}}(S', a') \right]$

- Note: $\mathbb{E}$ is not taken over randomness in $S'$ but is just w.r.t. the distribution returned by $z_{\bar{\theta}}$ (so $A^*$ is a RV)
Distributional RL

C51 Algorithm

- C51 = DQN, but $z_\theta$ instead of $q_\theta$; train $z_\theta$ by taking stochastic gradient descent steps on loss:
  
  \[ KL \left( \delta(R) + \gamma z_{\bar{\theta}}(S', A^*) \left\| z_\theta(S, A) \right. \right) \]

  with $(S, A, R, S')$ from replay buffer and $A^* = \arg \max_a \mathbb{E} \left[ z_{\bar{\theta}}(S', a') \right]$

- Note: we have replaced $z_R(S, A)$ with point mass $\delta(R)$
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C51 Algorithm

$$KL\left( \delta(R) + \gamma z_{\bar{\theta}}(S', A^*) \right\| z_{\theta}(S, A) \)$$
Distributional RL

C51 Algorithm

$$KL\left( \delta(R) + \gamma z_{\hat{\theta}}(S', A^*) \parallel z_{\theta}(S, A) \right)$$
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C51 Algorithm

\[ KL \left( \delta(R) + \gamma z_{\tilde{\theta}}(S', A^*) \right) \| z_{\theta}(S, A) \]
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C51 Algorithm
Current Research Topics

- Still an active area of research to understand why learning distributions of values helps when all we care about is the expectation
  - Framework for inductive bias?
  - Better-behaved optimisation?
- How can we efficiently optimise the Wasserstein loss instead of the KL divergence?
  - Theory motivates the Wasserstein metric, but empirically, stochastic approximations of Wasserstein loss perform very poorly
Questions?

\[ KL\left( \delta(R) + \gamma z_{\theta}(S', A^*) \right) \parallel z_{\theta}(S, A) \]